The Mean-Variance Anatomy of Behavioral Portfolios*

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Abstract

We study portfolio choice under generalized disappointment aversion in a static, model-free, multi-asset setting with short horizons. Despite nonlinear and kinked preferences, optimal portfolios admit a closed-form mean-variance representation with endogenously distorted moments. Risk-taking decomposes into standard and downside-driven components, revealing that disappointment aversion increases risk avoidance with horizon—reversing standard predictions. Ignoring downside asymmetries leads to a substantial welfare loss. Our framework delivers tractable portfolio design under behavioral preferences, offers testable implications, and highlights how psychological frictions reshape risk attitudes at short horizons before gradually fading as Sharpe ratios, skewness, and kurtosis regain dominance over longer horizons.

Keywords: Disappointment, mean-variance optimization, behavioral finance, downside risk, portfolio choice, investment horizon, risk-adjusted performance, non-expected utility.

JEL Classification: G11, D81, G12, C63, D14

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1 Introduction

Substantial evidence suggests that people are more motivated to avoid regret than to pursue glory. Allais (1953) famously challenged the expected utility theory by highlighting its inconsistency with the observed behavior: People frequently purchase insurance to protect against improbable losses while simultaneously buying lottery tickets to pursue improbable gains. This paradox reflects a tendency to overestimate small probabilities, fearing rare but painful outcomes, while also chasing outsized rewards. Such behavioral asymmetries have inspired alternative preference models, including prospect theory (Kahneman and Tversky; 1979) and disappointment aversion (DA) preferences (Gul; 1991), which were later generalized by Routledge and Zin (2010).

Generalized disappointment aversion (GDA) preferences allow investors to evaluate outcomes relative to an endogenous reference point, typically proportional to their certainty equivalent, and to assign greater weight to disappointing outcomes. These preferences have proven valuable in explaining several prominent asset pricing puzzles in equity markets (Campanale et al.; 2010; Bonomo et al.; 2011; Schreindorfer; 2019) and fixed income markets (Augustin and Tédongap; 2016, 2021), non-participation in the stock market (Ang et al.; 2005), and cross-sectional return anomalies (Delikouras; 2017; Farago and Tédongap; 2018; Delikouras and Kostakis; 2019). They also explain puzzling portfolio patterns that cannot be rationalized under standard utility frameworks (Dahlquist et al.; 2016).

This paper develops a novel theoretical and empirical framework for portfolio selection under GDA preferences in a multi-asset setting. Our approach departs from much of the existing literature by avoiding parametric assumptions on asset return dynamics and instead focusing on a model-free, single-period, buy-and-hold environment. Building on the work of Brandt (1999) and Ait-sahalia and Brandt (2001), we use high-frequency (daily) data to estimate optimal portfolios using the generalized method of moments (GMM), thereby allowing precise inference about tail-dependent quantities that are otherwise difficult to assess at low frequencies.

Our key theoretical contribution is to show that optimal portfolio choice under GDA preferences can be reformulated within a nonstandard mean-variance (MV) framework, where both the mean and variance are endogenously adjusted to reflect downside risk. To achieve this, we derive a fixed-point representation that yields an approximate closed-form solution using improved yet standard density approximations (see Yogo; 2006; Tédongap; 2015) within the moment condition. This structure produces three preference-adjusted quantities: an endogenous mean vector, an endogenous covariance matrix, and an endogenous risk aversion coefficient—a weighted average of standard and downside-specific risk aversions. In this GDA-augmented MV setting, the investor allocates wealth to the risk-free asset, the standard MV-efficient fund, and a specific GDA fund that hedges the risk of disappointment. This latter fund further decomposes into three distinct components: (i) a pure disappointment-hedging fund, (ii) a correction to the MV-efficient fund based on asset-specific downside (co)variances, and (iii) a corresponding adjustment to the hedging component. These funds reflect the mechanisms through which psychological frictions shape asset demand. Crucially, this decomposition is fully endogenous and varies between investors, which excludes a universal fund separation theorem under GDA, unlike models with more restrictive return structures (e.g., Simaan; 1993; Dahlquist et al.; 2016; Tédongap and Tinang; 2022).

To facilitate cross-investor comparisons, we introduce a novel concept of *effective risk aversion*—the level of risk aversion that would yield the endogenous mean-variance certainty equivalent in a standard MV framework evaluated at the GDA investor's optimal portfolio. This enables the recasting of GDA investor preferences into a standard mean-variance framework and provides a helpful benchmark for comparing how disappointment aversion reshapes investor behavior across different preference specifications and horizons.

We evaluate our theory using daily data from 1989 to 2022 across five major asset classes: cash, equity, commodity, bond, and real estate. Our empirical benchmark analysis focuses on short investment horizons ranging from one to twenty trading days, a setting rarely examined in the portfolio choice literature. While most studies analyze monthly or longer horizons and rely on predictive return dynamics (e.g., Brandt and Santa-Clara; 2006; Pastor and Stambaugh; 2012), our framework captures investor behavior over short holding periods using only observed return distributions.

Equity investing is often said to reward patience, with risk believed to decline over longer holding periods. Yet, the high cost of long-dated downside protection—such as deep out-of-the-money put options—tells a different story (Ralfe; 2024). While this disconnect has fueled debate over long-horizon risk, portfolio choice at short horizons—ranging from one day to a few weeks—has received surprisingly little attention despite its central role in risk monitoring, regulatory capital assessments, and the design of short-dated products. These horizons are also where behavioral frictions may be most acute: investors face frequent performance evaluations and exhibit heightened sensitivity to losses, which can amplify the effects of disappointment aversion. Moreover, return distributions at short horizons deviate sharply from normality, with pronounced skewness and kurtosis, further motivating downside-sensitive preference models. Our paper fills this gap by analyzing short-horizon asset allocation under GDA in a model-free setting, highlighting how psychological frictions reshape risk-taking behavior over time.

Even without explicit rebalancing or intertemporal motives, investors exhibit strong sensitivity to short-term disappointment risk. This complements dynamic models and suggests that behavioral preferences can generate meaningful horizon effects—even in a static setting with no assumptions about the return dynamics.

Our fixed-point approximation yields highly accurate results, capturing more than 98% of the true certainty equivalent across all horizons, even under strong aversion and asymmetry. When the disappointment threshold coincides with the certainty equivalent (i.e., symmetric disappointment), investors withdraw entirely from risky assets once the downside penalty exceeds a horizon-specific threshold—generalizing the nonparticipation result of Ang et al. (2005) to a multi-asset, model-free context. This withdrawal point rises from just 0.10 at a one-day horizon to 0.70 at twenty days.

The welfare loss from ignoring the downside asymmetries is substantial. For investors with moderate disappointment aversion and short investment horizons, relying on a standard mean-variance strategy leads to a welfare loss exceeding 7% of their certainty equivalent. This gap widens with a stronger aversion to disappointment and a more nuanced understanding of the definition of disappointment. Standard mean-variance portfolios sometimes lose up to 14% of certainty equivalent value, highlighting the importance of hedging downside risk.

Interestingly, GDA investors who fear falling short of their certainty equivalent—say, by at least 2.5%—and those disappointed unless outcomes exceed it by the same margin exhibit symmetric but opposite forms of disappointment perception. However, the behavior is not equally sensitive in both directions: aversion to downside losses outweighs the desire for upside gains, consistent with previous findings. As a result, holding other factors constant, the certainty equivalent of the downside-focused investor converges more slowly to that of the expected utility (EU) benchmark as the disappointment threshold shifts away from the certainty equivalent. This asymmetry reveals a fundamental skew in the way disappointment influences risk attitudes.

A central empirical finding of our study is that, under GDA preferences and short horizons, investors' willingness to assume risk *declines* with the investment horizon. For example, a representative investor who experiences disappointment when returns fall at least 2.5% below the certainty equivalent reduces their stock allocation from 46% to 27% and their bond allocation from 124% to 55% as the horizon extends from one to twenty trading days. This pattern sharply contrasts with the dominant predictions of dynamic portfolio theory at longer horizons, where risk-taking typically increases with horizon due to return mean reversion or intertemporal hedging motives (e.g., Merton; 1969; Campbell and Viceira; 2002; Gollier; 2002). When returns are predictable—often through the price-dividend ratio—standard expected utility models imply that the variance of cumulative returns grows more slowly than the mean, improving the long-term risk-return trade-off. This implication is widely echoed in empirical findings and investment advice (e.g., Bogle; 1994; Malkiel; 1996; Ameriks and Zeldes; 2004; Calvet and Campbell; 2009).

However, some exceptions challenge this conventional view. Barberis (2000) shows that incorporating parameter uncertainty into an otherwise i.i.d. return environment can reduce long-term equity exposure. Similarly, Gollier and Zeckhauser (2002) demonstrate that, under expected utility and complete markets, longer horizons increase risk-taking only if absolute risk tolerance is convex—highlighting that a rising risk profile with the horizon is not a universal result. Without such restrictive assumptions, our findings show that Generalized Disappointment Aversion reverses the standard horizon-risk logic at short horizons. For investors who experience disappointment when outcomes fall below a benchmark lower than their certainty equivalent, the perceived probability of loss rises with the horizon, amplifying the psychological cost of risk. Conversely, for investors disappointed unless outcomes exceed a benchmark above their certainty equivalent, the perceived value of upside gains diminishes over time due to temporal averaging. In both cases, GDA preferences induce a nonstandard horizon-risk profile marked by increasing conservatism as the horizon lengthens—even in a static, model-free environment—underscoring the pivotal role of psychological frictions in shaping optimal portfolio allocations.

We note, however, that this pattern does not emerge under symmetric disappointment preferences—i.e., when the disappointment threshold equals the certainty equivalent, as in Gul (1991). In such cases, and when disappointment aversion is sufficiently low, both the certainty equivalent and optimal risk exposure increase with the investment horizon, consistent with standard predictions. Therefore, the decreasing horizon-risk relationship we document is a distinctive feature of the GDA framework of Routledge and Zin (2010), which allows for asymmetric disappointment thresholds and captures richer behavioral responses across horizons.

While our approach generalizes to longer horizons, the approximation becomes less reliable as the horizon increases, and the endogenous mean-variance structure no longer holds the same level of accuracy. Nonetheless, extending the exact numerical solution up to several months reveals a striking U-shaped risk profile: investors initially reduce risk exposure with horizon but eventually increase it as standard performance metrics—such as improving kurtosis, less negative skewness, and higher annualized Sharpe ratios—begin to dominate behavioral frictions. These results highlight that psychological asymmetries drive conservative behavior at short horizons but fade in importance over time. Hence, our main insights and most robust contributions concern the short end of the horizon spectrum, where disappointment aversion reshapes risk attitudes and generates nonstandard portfolio behavior.

The remainder of the article is organized as follows. Section 2 presents the theoretical portfolio choice framework, which features a model-free, static environment with a fixed investment horizon and generalized disappointment aversion preferences. We derive the first-order condition for optimal asset allocation and show how this leads to a nonstandard mean-variance representation under regular and downside density approximations. This formulation admits a dual interpretation. In addition, it defines an effective risk aversion coefficient under GDA and enables a tractable decomposition of the optimal portfolio. Section 3 elaborates on this decomposition, distinguishing between common and investor-specific components and interpreting their economic functions. Section 4 describes the data, summarizes key empirical moments, and reports the main results, including certainty equivalent comparisons, portfolio weights, approximation accuracy, hedging benefits, and horizon effects. Section 5 concludes. An internal appendix provides analytical derivations supporting the main results. An online appendix extends the analysis to include performance evaluation, illustrating how our GDA-based framework informs practical asset and portfolio performance measurement. It also provides additional tables and figures to complement the empirical findings.

2 Theoretical Setup

We consider an economy with $n+1$ assets, with $n \geq 1$. An investor can allocate her wealth between n risky securities ($i = 1, 2, \ldots, n$) and a risk-free asset ($i = 0$). Similarly to Ang and Bekaert (2002), Das and Uppal (2004), Ang et al. (2005), Guidolin and Timmermann (2008), and Dahlquist et al. (2016), we consider a finite-horizon setup with the utility defined over the terminal wealth. Our framework is set in discrete time. Let $R = (R_1, R_2, \ldots, R_n)^\top$ denote the vector of the simple gross returns on risky assets, where R_i is the return of the asset i, and let R_0 denote the simple gross risk-free return. Throughout the article, we will make no specific assumptions about the distribution of returns. Our theoretical findings are independent of the distribution of returns, and our empirical findings are based on their realization over a historical sample.

A portfolio strategy can be described by a $n \times 1$ vector $w = (w_1, w_2, \ldots, w_n)^\top$ where w_i is the portfolio weight in the risky asset i. The simple gross return of the portfolio strategy w is given by

$$R_w = R_0 + w^\top (R - \mathbf{1}R_0), \tag{1}$$

where $\mathbf{1}$ denotes the n-dimensional vector of ones. We assume no short-selling or borrowing constraints, so we have $w \in \mathbb{R}^n$. We aim to characterize, both theoretically and empirically, the optimal value of w that an investor chooses to maximize a given welfare objective. In this article, the objective is considered the certainty equivalent of the investor's terminal wealth over a given investment horizon, defined by the investor's attitude towards risk.

2.1 Investor's attitude towards risk and optimal portfolio solution

We consider an investor whose objective at the initial date (say, 0), starting with the initial wealth W_0, is to maximize the certainty equivalent of the terminal wealth W_τ, where τ is the final date, i.e., the investment horizon. The terminal wealth may be written

$$W_\tau = W_0 R_{w,\tau} \tag{2}$$

where $R_{w,\tau}$ is the wealth simple gross return over the investment horizon τ, and w is the investor's buy-and-hold portfolio strategy over the investment horizon. So, $R_{w,\tau}$ is the simple gross return on the investor's portfolio.

We assume the investor has generalized disappointment aversion (GDA) preferences, as in Routledge and Zin (2010). GDA preferences have the desirable property that investors care differently about downside losses than they do about upside gains. A disappointment-averse investor is loss-averse around an endogenous reference point proportional to her certainty equivalent. Following Routledge and Zin (2010), for the GDA investor, the certainty equivalent of terminal wealth, $\mathcal{C}\left(W_\tau\right)$, is implicitly defined by

$$\eta U\left(\mathcal{C}\left(W_\tau\right)\right)=\mathbb{E}\left[U\left(W_\tau\right)\right]-\ell \mathbb{E}\left[\left(U\left(\kappa \mathcal{C}\left(W_\tau\right)\right)-U\left(W_\tau\right)\right) \mathbb{I}\left(W_\tau<\kappa \mathcal{C}\left(W_\tau\right)\right)\right], \tag{3}$$

where $\mathbb{I}(\cdot)$ is an indicator function that equals one if the condition is met and zero otherwise, and

$$U(x)= \begin{cases}\dfrac{x^{1-\gamma}}{1-\gamma} & \text { if } \gamma>0 \text { and } \gamma \neq 1, \\ \ln x & \text { if } \gamma=1.\end{cases} \tag{4}$$

The parameter $\gamma>0$ measures the investor's risk aversion, $\ell \geq 0$ is the investor's degree of disappointment aversion, and $\kappa>0$ is the percentage of her certainty equivalent below which outcomes are considered disappointing. The parameter η is not free and is defined as

$$\eta=1-\ell\left(\kappa^{1-\gamma}-1\right) \mathbb{I}(\kappa>1), \tag{5}$$

and ensures that the certainty equivalent is appropriately scaled even when $\kappa>1$ so that the certainty equivalent of a constant value, x, equals itself (i.e., $\mathcal{C}(x)=x$).

If the investor's degree of disappointment aversion is zero ($\ell=0$), the definition of the certainty equivalent from (3) simplifies to

$$U\left(\mathcal{C}\left(W_\tau\right)\right)=\mathbb{E}\left[U\left(W_\tau\right)\right]. \tag{6}$$

In this case, the investor has expected utility (EU) preferences with the power utility. Throughout the paper, we refer to such an investor as the EU investor. When $\ell>0$, outcomes lower than $\kappa \mathcal{C}\left(W_\tau\right)$ receive an extra weight and lower the investor's certainty equivalent relative to EU. To maximize the certainty equivalent, a disappointment-averse investor would like to avoid outcomes below $\kappa \mathcal{C}\left(W_\tau\right)$. The penalty for disappointing outcomes increases with ℓ, so this parameter modulates the importance of disappointment versus satisfaction and can be interpreted as the degree of disappointment aversion.

The parameter κ defines the threshold for disappointing outcomes relative to the certainty equivalent. The special case of $\kappa = 1$ corresponds to the original disappointment aversion (DA) preferences of Gul (1991). If $\kappa < 1$, the random future value is considered disappointing if it lies sufficiently below today's certainty equivalent; if $\kappa > 1$, the random future value must be sufficiently far above the certainty equivalent to be considered not disappointing. Previous literature on disappointment aversion primarily concerns asset pricing and the $\kappa < 1$ case. Routledge and Zin (2010) briefly discuss the $\kappa > 1$ possibility also in an asset pricing context,[1] but otherwise, the literature has mainly ignored this setting. One exception is Dahlquist et al. (2016), who show in a portfolio choice context that an investor with $\kappa > 1$ endogenously displays a preference for negative asymmetry in asset returns. As pointed out by Dahlquist et al. (2016), the setting where the reference point is lower than the certainty equivalent is arguably more relevant for understanding real-life investor behavior. However, it might be of general interest to study the portfolio choice implications of a setting where the reference point is higher than the certainty equivalent. We follow these authors and demonstrate that different values of κ lead to diverse investor behavior. Similarly, we refer to an investor with $\kappa = 1$ as a DA investor and an investor with $\kappa \neq 1$ as a GDA investor.

Due to the homogeneity of utility function (4), we have

$$\mathcal{C}\left(W_\tau\right) = W_0 \mathcal{C}\left(R_{w,\tau}\right). \tag{7}$$

Ultimately, the investor's objective is to maximize the certainty equivalent of the portfolio gross return, $\mathcal{C}\left(R_{w,\tau}\right)$, given by

$$\eta U\left(\mathcal{C}_\tau\right) = \mathbb{E}\left[U\left(R_{w,\tau}\right)\right] - \ell \mathbb{E}\left[\left(U\left(\kappa \mathcal{C}_\tau\right) - U\left(R_{w,\tau}\right)\right) \mathbb{I}\left(R_{w,\tau} < \kappa \mathcal{C}_\tau\right)\right], \tag{8}$$

in which we have used the short-hand notation $\mathcal{C}_\tau$ for $\mathcal{C}\left(R_{w,\tau}\right)$. We show in Appendix A that the first-order condition for maximizing the certainty equivalent $\mathcal{C}_\tau$ with respect to w

[1] In an intertemporal consumption-based general equilibrium asset pricing model, Routledge and Zin (2010) discuss the value of this parameter in connection with the autocorrelation of consumption growth modeled as a simple two-state Markov chain. To generate counter-cyclical risk aversion, they state that a value less than one for κ is needed when there is a negative autocorrelation of consumption growth and a value greater than one when the autocorrelation is positive. The economic mechanism behind this link is the substitution effect.

is given by the following Euler equation:

$$\mathbb{E}\left[R_{w,\tau}^{-\gamma}\left(1+\ell\mathbb{I}\left(\mathcal{D}_\tau\right)\right)R_\tau^e\right]=0, \tag{9}$$

where $\mathcal{D}_\tau$ defines the investor's disappointing event, i.e., $\mathcal{D}_\tau \equiv \left\{R_{w,\tau} < \kappa\mathcal{C}_\tau\right\}$, and R_τ^e defines the vector of asset excess returns over the risk-free rate, i.e., $R_\tau^e \equiv R_\tau - \mathbf{1}R_{0,\tau}$. It appears that equations (9) and (8) must be solved simultaneously for the optimal allocation w and the optimal certainty equivalent $\mathcal{C}_\tau$. Formally, given asset returns, investor's preference parameters, and the allocation w, equation (8) is solved for the certainty equivalent $\mathcal{C}_\tau$ which is then substituted into equation (9). At this point, equation (9) is a system of n equations with n unknowns that are the elements of the vector w. Since the system is expressed in the form of moment conditions, it appears that given the stationary asset returns historical time series data and the investor's preference parameters, w is the vector parameter estimate of an identified GMM system. This approach of solving an asset allocation problem without imposing any parametric structure on the asset return dynamics is developed in Brandt (1999) and Ait-sahalia and Brandt (2001).

2.2 Optimal portfolio characterization

Standard mean-variance preferences in a portfolio choice model represent the preferences of an investor who evaluates alternative portfolios based on the mean and the variance of returns. As shown previously, the optimal portfolio w in our GDA setting is a solution to a nonlinear system of equations. It would be interesting to characterize this solution further to better understand the different return attributes on which investors with non-standard GDA preferences evaluate alternative portfolios.

We show in Appendix C that the above first-order condition for optimality, i.e., equation (9), can equivalently be written as follows:

$$\mathbb{E}\left[\frac{R_{w,\tau}^{-\gamma}}{\mathbb{E}\left[R_{w,\tau}^{-\gamma}\right]}R_\tau^e\right]+\ell\pi_{\mathcal{D},\tau}^{*}\mathbb{E}\left[\frac{R_{w,\tau}^{-\gamma}}{\mathbb{E}\left[R_{w,\tau}^{-\gamma}\mid\mathcal{D}_\tau\right]}R_\tau^e\mid\mathcal{D}_\tau\right]=0 \tag{10}$$

with

$$\pi^*_{\mathcal{D},\tau} \equiv \frac{\mathbb{E}\left[R_{w,\tau}^{-\gamma}\mathbb{I}\left(\mathcal{D}_\tau\right)\right]}{\mathbb{E}\left[R_{w,\tau}^{-\gamma}\right]} = \mathbb{E}\left[\frac{R_{w,\tau}^{-\gamma}}{\mathbb{E}\left[R_{w,\tau}^{-\gamma}\right]}\mathbb{I}\left(\mathcal{D}_\tau\right)\right] = \mathbb{E}^*\left[\mathbb{I}\left(\mathcal{D}_\tau\right)\right] = \text{prob}^*\left(\mathcal{D}_\tau\right), \tag{11}$$

where $\mathbb{E}^*[\cdot]$ denotes the distorted expectation operator induced by the change-of-measure $\frac{R_{w,\tau}^{-\gamma}}{\mathbb{E}\left[R_{w,\tau}^{-\gamma}\right]}$, and $\text{prob}^*(\cdot)$ denotes the associated distorted probability operator. In fact, since the term $\frac{R_{w,\tau}^{-\gamma}}{\mathbb{E}\left[R_{w,\tau}^{-\gamma}\right]}$ is positive and has an expectation equal to unity, it can be considered as distorting the original probability distribution of the asset returns. As discussed by Hansen et al. (2008), this distortion indicates a rather different interpretation of the parameter γ. Following Anderson et al. (2003), the parameter γ may reflect the investor's concerns about not knowing the precise riskiness that she confronts in the marketplace. In this case, the original probability distribution of asset returns is viewed as a statistical approximation, and the investor is concerned that it may be misspecified. Therefore, we refer to $\pi^*_{\mathcal{D},\tau}$ as the beliefs-implied disappointment probability, that is, the disappointment probability as perceived by the investor based on her beliefs.

From equation (10), we consider the following density approximations:

$$\begin{aligned} \frac{R_{w,\tau}^{-\gamma}}{\mathbb{E}\left[R_{w,\tau}^{-\gamma}\right]} &\approx 1 - \gamma_\tau\left(R_{w,\tau}^e - \mathbb{E}\left[R_{w,\tau}^e\right]\right) \\ \frac{R_{w,\tau}^{-\gamma}}{\mathbb{E}\left[R_{w,\tau}^{-\gamma} \mid \mathcal{D}_\tau\right]} &\approx 1 - \gamma_{\mathcal{D},\tau}\left(R_{w,\tau}^e - \mathbb{E}\left[R_{w,\tau}^e \mid \mathcal{D}_\tau\right]\right), \end{aligned} \tag{12}$$

where

$$\gamma_\tau = \frac{\sigma\left[R_{w,\tau}^{-\gamma}\right]}{\mathbb{E}\left[R_{w,\tau}^{-\gamma}\right]} \Big/ \sigma\left[R_{w,\tau}^e\right] \quad \text{and} \quad \gamma_{\mathcal{D},\tau} = \frac{\sigma\left[R_{w,\tau}^{-\gamma} \mid \mathcal{D}_\tau\right]}{\mathbb{E}\left[R_{w,\tau}^{-\gamma} \mid \mathcal{D}_\tau\right]} \Big/ \sigma\left[R_{w,\tau}^e \mid \mathcal{D}_\tau\right]. \tag{13}$$

As previously discussed, the first density in equation (12) distorts the original probability distribution of asset returns, while the corresponding approximation preserves the mean and volatility.[2] Likewise, the second density distorts the probability distribution of asset returns conditional on the disappointing event, and the corresponding approximation preserves the mean and the volatility conditional on the disappointing event. The expression for γ_τ is appealing. It is the ratio of two quantities. The numerator reminds us of the Hansen-Jagannathan bound and the denominator is the portfolio standard deviation. Indeed, if $\ell = 0$ so that the investor does not care about downside risk, the numerator in the expression for γ_τ in equation (13) represents the maximum Sharpe ratio attainable by the investor. Dividing the Sharpe ratio by the portfolio standard deviation yields the ratio of the expected excess return to the variance, a measure of risk aversion as typically understood in the asset pricing literature. Based on this observation, we refer to γ_τ as the investor's standard level of risk aversion. By analogy, we refer to $\gamma_{\mathcal{D},\tau}$ as the investor's aversion to downside risk.

Substituting out the density approximations (12) into equation (10) and rearranging, we show in Appendix C that the optimal portfolio may be expressed as the following fixed-point equation:

$$w = \frac{1}{\tilde{\gamma}_\tau} \tilde{\Sigma}_\tau^{-1} \tilde{\mu}_\tau \tag{14}$$

$$\text{where} \quad \begin{cases} \tilde{\Sigma}_\tau = \left(1 - \alpha_{\mathcal{D},\tau}\right) \Sigma_\tau + \alpha_{\mathcal{D},\tau} \Sigma_{\mathcal{D},\tau} \quad \text{and} \quad \tilde{\mu}_\tau = \left(1 - \lambda_{\mathcal{D},\tau}\right) \mu_\tau + \lambda_{\mathcal{D},\tau} \mu_{\mathcal{D},\tau} \\ \text{with} \quad \lambda_{\mathcal{D},\tau} = \dfrac{\ell \pi^*_{\mathcal{D},\tau}}{1 + \ell \pi^*_{\mathcal{D},\tau}}, \quad \tilde{\gamma}_\tau = \left(1 - \lambda_{\mathcal{D},\tau}\right) \gamma_\tau + \lambda_{\mathcal{D},\tau} \gamma_{\mathcal{D},\tau} \quad \text{and} \quad \alpha_{\mathcal{D},\tau} = \dfrac{\lambda_{\mathcal{D},\tau} \gamma_{\mathcal{D},\tau}}{\tilde{\gamma}_\tau}, \end{cases}$$

where $\mu_\tau = \mathbb{E}\left[R^e_\tau\right]$ and $\Sigma_\tau = \text{var}\left[R^e_\tau\right]$ are respectively the mean vector and the variance-covariance matrix of risky asset excess returns, and $\mu_{\mathcal{D},\tau} = \mathbb{E}\left[R^e_\tau \mid \mathcal{D}_\tau\right]$ and $\Sigma_{\mathcal{D},\tau} = \text{var}\left[R^e_\tau \mid \mathcal{D}_\tau\right]$ are respectively the mean vector and the variance-covariance matrix of risky asset excess returns conditional on the disappointing event. The coefficients $\tilde{\gamma}_\tau$, $\alpha_{\mathcal{D},\tau}$, and $\lambda_{\mathcal{D},\tau}$, all depend on the optimal portfolio weight vector, w (i.e., they are endogenously determined). Likewise, the downside event $\mathcal{D}_\tau$, as well as the downside variance-covariance matrix of the assets, $\Sigma_{\mathcal{D},\tau}$, and the downside expected excess returns of the assets, $\mu_{\mathcal{D},\tau}$, are all endogenous, therefore justifying why downside risk is said to be endogenous in this setting.

[2] Density approximations similar to equation (12) are used elsewhere in the literature, for example to derive linear cross-sectional implications of asset pricing models (see for example Yogo; 2006, and Farago and Tédongap; 2018). These authors however generally match only the unitary mean of the density, and this would mean $\gamma_\tau = \gamma$, while we also match its volatility. Matching the volatility of investors' beliefs can be crucial in certain applications, such as determining the optimal size of the equity premium.

Equation (14) is finally a fixed-point equation that is equivalent to the Euler equation (9) under density approximations (12). To solve for the optimal portfolio under these conditions, without imposing any parametric structure on the asset return dynamics, we proceed as follows. Given historical time series data on asset returns, investor preference parameters, and allocation w, we replace expectation operators in equation (8) by sample moments and solve for the equivalent certainty $\mathcal{C}_\tau$ which is then substituted in equation (14). At this point, (14) is a nonlinear fixed-point equation in the portfolio weight w, and we solve for w after replacing the population standard and downside first and second moments of asset returns by their sample counterparts. To assess the impact of the density approximations (12) on the portfolio solution, we will compare the certainty equivalent achieved by the solution obtained via the Euler equation (9) to the certainty equivalent achieved by the solution obtained through the fixed-point equation (14).

Equation (14) suggests that GDA investors care not only about the standard mean and variance-covariance of asset returns but also their downside mean and variance-covariance. This latter observation leads to an elegant interpretation of the GDA portfolio choice problem in a mean-variance framework.

Weighted-average mean-variance interpretation. Given the endogenous values of γ_τ, $\gamma_{\mathcal{D},\tau}$ and $\lambda_{\mathcal{D},\tau}$, and given the endogenous downside expected returns vector $\mu_{\mathcal{D},\tau}$ and downside variance-covariance matrix $\Sigma_{\mathcal{D},\tau}$ of the risky asset returns, it is interesting to observe that the optimal allocation (14) can also be achieved by solving the following weighted-average mean-variance (WAMV) investment problem:

$$\max_{w} \; \left(1-\lambda_{\mathcal{D},\tau}\right)\left(\mu_{w,\tau}-\frac{\gamma_\tau}{2}\sigma^2_{w,\tau}\right)+\lambda_{\mathcal{D},\tau}\left(\mu_{w\mathcal{D},\tau}-\frac{\gamma_{\mathcal{D},\tau}}{2}\sigma^2_{w\mathcal{D},\tau}\right), \tag{15}$$

where $\mu_{w,\tau} = w^\top \mu_\tau$ and $\sigma^2_{w,\tau} = w^\top \Sigma_\tau w$ are the standard expected excess return and the standard variance of the investor's portfolio, while $\mu_{w\mathcal{D},\tau} = w^\top \mu_{\mathcal{D},\tau}$ and $\sigma^2_{w\mathcal{D},\tau} = w^\top \Sigma_{\mathcal{D},\tau} w$ are the downside expected excess return and the downside variance of the investor's portfolio. In particular, equation (15) validates our previous interpretation of the coefficients γ_τ and $\gamma_{\mathcal{D},\tau}$. In forming the WAMV certainty equivalent that she maximizes for the optimal portfolio, the investor assigns the weight $1-\lambda_{\mathcal{D},\tau}$ to the standard mean-variance certainty equivalent $\mu_{w,\tau} - \frac{\gamma_\tau}{2}\sigma^2_{w,\tau}$ in which γ_τ is interpretable as the investor's standard risk aversion as usually understood. The remaining weight $\lambda_{\mathcal{D},\tau}$ is assigned to what we refer to as the downside mean-variance certainty equivalent $\mu_{w\mathcal{D},\tau} - \frac{\gamma_{\mathcal{D},\tau}}{2}\sigma^2_{w\mathcal{D},\tau}$ in which, by analogy, $\gamma_{\mathcal{D},\tau}$ is interpretable as the investor's downside risk aversion. The weight $\lambda_{\mathcal{D},\tau}$ assigned to the downside mean-variance certainty equivalent reflects the investor's degree of disappointment aversion ℓ and her endogenous perception for downside risk $\pi^*_{\mathcal{D},\tau}$.

Endogenous mean-variance interpretation. Ultimately, another interpretation of the optimal portfolio solution in equation (14) is that a GDA investor endogenously behaves just like a mean-variance investor. This is because, given the endogenous quantities $\tilde{\gamma}_\tau$, $\tilde{\Sigma}_\tau$ and $\tilde{\mu}_\tau$, jointly implied by the GDA investor preferences and observed returns, the optimal portfolio in equation (14) can also be achieved by solving the following endogenous mean-variance investment problem

$$\max_w \ \tilde{\mu}_{w,\tau} - \frac{\tilde{\gamma}_\tau}{2}\tilde{\sigma}^2_{w,\tau}, \tag{16}$$

where we refer to $\tilde{\mu}_{w,\tau} = w^\top \tilde{\mu}_\tau$ and $\tilde{\sigma}^2_{w,\tau} = w^\top \tilde{\Sigma}_\tau w$ as the endogenous expected excess return and the endogenous variance of the investor's portfolio, respectively. It is straightforward from equation (14) that, for any asset in the economy and any portfolio, the endogenous expected excess return is a weighted average of the standard expected excess return and the downside expected excess return, with the weights given by $1 - \lambda_{\mathcal{D},\tau}$ and $\lambda_{\mathcal{D},\tau}$, respectively. Similarly, the endogenous variance is a weighted average of the standard variance and the downside variance, with the weights given by $1 - \alpha_{\mathcal{D},\tau}$ and $\alpha_{\mathcal{D},\tau}$, respectively. Likewise, we refer to $\tilde{\gamma}_\tau$ as the investor's endogenous risk aversion. Therefore, equation (14) shows that the investor's endogenous risk aversion is a weighted average of her standard risk aversion and her downside risk aversion, with the weights given by $1 - \lambda_{\mathcal{D},\tau}$ and $\lambda_{\mathcal{D},\tau}$, respectively. Henceforth, the portfolio $\tilde{\Sigma}^{-1}_\tau \tilde{\mu}_\tau$ will be called the investor's endogenous mean-variance fund, as it endogenously depends on the specific investor's GDA preference parameters.

If $\ell = 0$, i.e. for EU investors, we have $\lambda_{\mathcal{D},\tau} = 0$ and equation (15) implies that an EU investor behaves like a typical mean-variance investor who cares only about the standard expected excess return and the standard variance of a portfolio, consistent with the results of Levy and Markowitz (1979) and Hlawitschka (1994). This result is also confirmed by Dahlquist et al. (2016), who find that the non-normality of asset returns only has a marginal effect on the optimal portfolios of EU investors. Observe from equation (14) that $\lambda_{\mathcal{D},\tau} = 0$ is equivalent to $\alpha_{\mathcal{D},\tau} = 0$, which implies $\tilde{\Sigma}_\tau = \Sigma_\tau$ and $\tilde{\mu}_\tau = \mu_\tau$. The investor optimally allocates her wealth between the risk-free asset and the standard mean-variance efficient fund $\Sigma_\tau^{-1}\mu_\tau$. This pair of funds is identical for all EU investors, and the share of wealth invested in the risky fund reflects the investor's endogenous risk aversion, which appears to be equal to her standard risk aversion (i.e., $\tilde{\gamma}_\tau = \gamma_\tau$).

To the contrary of an EU investor who assesses the portfolio reward through the standard expected excess return $\mu_{w,\tau}$ and the portfolio risk through the standard variance $\sigma^2_{w,\tau}$, the GDA investor, in addition, cares about the downside expected excess return $\mu_{w\mathcal{D},\tau}$ and the downside variance $\sigma^2_{w\mathcal{D},\tau}$ of a portfolio. A GDA investor assesses the reward and the risk of a portfolio through its endogenous expected excess return $\tilde{\mu}_{w,\tau}$ and its endogenous variance $\tilde{\sigma}^2_{w,\tau}$, respectively, each of which is a weighted average of their standard and downside counterparts. The weight assigned to the downside counterparts in the GDA investor's assessment of a portfolio reward and risk reflects the investor's degree of disappointment aversion and her endogenous perception of downside risk.

It would be inappropriate to compare two GDA investors with different sets of preference parameters based solely on their endogenous risk aversion ($\tilde{\gamma}_\tau$). This is because each investor evaluates portfolios using distinct endogenous mean vectors and covariance matrices of asset returns ($\tilde{\mu}_\tau, \tilde{\Sigma}_\tau$). Following Dahlquist et al. (2016), we suggest comparing investors by their "effective risk aversion", hereby defined as the risk aversion level $\tilde{\gamma}^e_\tau$ that allows them to achieve their endogenous MV certainty equivalent in a standard MV framework. Formally, $\tilde{\gamma}^e_\tau$ is derived by solving:

$$\mu_{w,\tau} - \frac{\tilde{\gamma}^e_\tau}{2}\sigma^2_{w,\tau} = \tilde{\mu}_{w,\tau} - \frac{\tilde{\gamma}_\tau}{2}\tilde{\sigma}^2_{w,\tau} \quad \text{where} \quad w = \frac{1}{\tilde{\gamma}_\tau}\tilde{\Sigma}_\tau^{-1}\tilde{\mu}_\tau. \tag{17}$$

In Appendix B, we prove that:

$$\tilde{\gamma}^e_\tau = \tilde{\gamma}_\tau \frac{\left(2\mu_\tau - \tilde{\mu}_\tau\right)^\top \tilde{\Sigma}_\tau^{-1}\tilde{\mu}_\tau}{\tilde{\mu}_\tau^\top \tilde{\Sigma}_\tau^{-1}\Sigma_\tau\tilde{\Sigma}_\tau^{-1}\tilde{\mu}_\tau}. \tag{18}$$

3 Optimal portfolio decomposition

Decomposing the portfolio strategy into its key components enhances our understanding of the different mechanisms through which an investor forms asset demands and how each component is tied to a particular attribute of the investor's attitude toward risk. We aim to provide an endogenous decomposition of the optimal portfolio.

Dahlquist et al. (2016) impose a specific structure for asset returns in the economy through a model in which asset returns are assumed to follow a multivariate normal-exponential distribution. They show that this particular framework leads to a three-fund separation strategy where a GDA investor optimally allocates wealth to the risk-free asset, the standard mean-variance efficient fund, and an additional fund reflecting return non-normality, which they term the asymmetry-variance efficient fund. This triplet of funds is identical for all GDA investors in their framework. The share of wealth invested in the standard mean-variance efficient fund reflects the investor endogenous risk aversion. In contrast, the share of wealth invested in the asymmetry-variance efficient fund additionally depends on the investor implicit asymmetry aversion. To the contrary of Dahlquist et al. (2016), in our current framework, we do not impose any particular model for asset returns.

We argue that a common fund separation strategy does not hold in our general and model-free setting. That is, we generally cannot find a k-tuple of funds that is identical for all GDA investors, so any GDA investor's optimal portfolio can be constructed by holding each of these k funds in appropriate ratios, where the number of funds is sufficiently smaller than the number of individual assets in the portfolio, i.e., $k << n$. The fund separation theorem does not hold in general since funds would be based on common investors' beliefs about expected excess returns and their variance-covariance matrix, both conditionally on the downside event or not. However, even though all investors in our setting would observe the same (μ_τ, Σ_τ), it remains that their $\left(\mu_{\mathcal{D},\tau}, \Sigma_{\mathcal{D},\tau}\right)$ are very likely to be different, since the downside event is endogenous and not identical for all investors, i.e., it varies with investor's preference parameters. Indeed, as shown in equation (14), the optimal portfolio satisfies a two-fund structure composed of the risk-free rate and the endogenous mean-variance fund $\tilde{\Sigma}_\tau^{-1}\tilde{\mu}_\tau$ which is not identical for all GDA investors as it endogenously depends on the investor's preference parameters. In what follows, we aim to dissect this investor's endogenous mean-variance fund to understand its formation fully.

In Appendix C, we develop from the expressions of $\tilde{\Sigma}_\tau$ and $\tilde{\mu}_\tau$ in equation (14) to prove that the GDA investor's optimal portfolio may be decomposed into four components as follows:

$$w = \frac{1}{\tilde{\gamma}_\tau}\left(w^{\mathbf{MV}} - \alpha_{\mathcal{D},\tau} w^{\mathbf{MVA}}\right) + \frac{\lambda_{\mathcal{D},\tau}}{\tilde{\gamma}_\tau}\left(w^{\mathbf{DH}} - \alpha_{\mathcal{D},\tau} w^{\mathbf{DHA}}\right) \tag{19}$$

with

$$
\begin{aligned}
w^{\mathbf{MV}} &= \Sigma_\tau^{-1}\mu_\tau \quad \text{and} \quad w^{\mathbf{MVA}} = A_{\mathcal{D},\tau} w^{\mathbf{MV}} \\
w^{\mathbf{DH}} &= \Sigma_\tau^{-1}\left(\mu_{\mathcal{D},\tau} - \mu_\tau\right) \quad \text{and} \quad w^{\mathbf{DHA}} = A_{\mathcal{D},\tau} w^{\mathbf{DH}} \\
\text{where} \quad A_{\mathcal{D},\tau} &= \left[\Sigma_\tau + \alpha_{\mathcal{D},\tau}\left(\Sigma_{\mathcal{D},\tau} - \Sigma_\tau\right)\right]^{-1}\left(\Sigma_{\mathcal{D},\tau} - \Sigma_\tau\right).
\end{aligned}
\tag{20}
$$

Equation (19) shows that GDA investors optimally invest into four risky funds in addition to the risk-free rate. More precisely, in addition to the risk-free rate and the standard mean-variance efficient fund common to all investors, three additional funds are tailor-made to the specific characteristics of the GDA investor. To the contrary of the EU investor, the GDA investor cares about downside risk. Therefore, the GDA investor positions in these specific funds allow her overall portfolio to deviate from the portfolio of an EU investor with equal endogenous risk aversion. The standard mean-variance efficient fund $w^{\mathbf{MV}}$ would typically be long (short) in assets with a positive (negative) risk premium, and the amount of the position in each asset depends on the ratio of the expected excess return to the variance (i.e., the reward per unit risk). Let us look at the remaining funds and how they alter the GDA investor's optimal portfolio relative to the EU investor.

The first investor's specific fund $w^{\mathbf{DH}}$ would typically be long (short) in assets with a positive (negative) relative downside expected excess return, where the relative downside expected excess return is the difference between the downside expected excess return and the standard expected excess return, i.e., elements of the vector $\mu_{\mathcal{D},\tau} - \mu_\tau$; the amount of the position in each asset depends on the ratio of the relative downside expected excess return to the variance. Notice that the relative downside expected excess return is otherwise related to the covariance between the excess return and the disappointing event, as we have

$$
\operatorname{cov}\left(R_\tau^e, \mathbb{I}\left(\mathcal{D}_\tau\right)\right) = \pi_{\mathcal{D},\tau}\left(\mu_{\mathcal{D},\tau} - \mu_\tau\right) \quad \text{with} \quad \pi_{\mathcal{D},\tau} = \mathbb{E}\left[\mathbb{I}\left(\mathcal{D}_\tau\right)\right] = \operatorname{prob}\left(\mathcal{D}_\tau\right). \tag{21}
$$

Therefore, the first investor's specific fund is equivalent to the portfolio $\Sigma_\tau^{-1}\operatorname{cov}\left(R_\tau^e, \mathbb{I}\left(\mathcal{D}_\tau\right)\right)$, i.e., the vector of population regression coefficients from a multiple regression of the disappointment indicator onto the set of risky asset excess returns. This latter characterization corresponds to a hedging demand, as is typical in portfolio choice literature. More precisely, the investor hedges the disappointing event by taking a long (short) position in assets that tend to rise (fall) when disappointment sets in. For this reason, we refer to this first investor's specific fund $w^{\mathbf{DH}}$ as the disappointment-hedging fund.

By premultiplying the vector $w^{\mathbf{MV}}$ by the matrix $A_{\mathcal{D},\tau}$, the second investor's specific fund $w^{\mathbf{MVA}}$ adjusts the asset positions in the mean-variance efficient fund by accounting for their relative downside variance (covariance), where the relative downside variance (covariance) is the difference between the downside variance (covariance) and the standard variance (covariance), i.e., elements of the matrix $\Sigma_{\mathcal{D},\tau} - \Sigma_\tau$. Therefore, we refer to this second investor's specific fund as the downside risk-adjusted mean-variance efficient fund. Since the weight that the investor assigns to this fund is negative as shown in equation (19), we argue that relative to an EU investor with equal endogenous risk-aversion, the GDA investor would typically reduce (increase) the amount of her standard mean-variance efficient fund position in assets which the downside variance is greater (lower) than the standard variance, i.e., the relative downside variance is positive (negative).

Likewise, by premultiplying the vector $w^{\mathbf{DH}}$ by the matrix $A_{\mathcal{D},\tau}$, the third and last investor's specific fund, $w^{\mathbf{DHA}}$, adjusts the asset positions in the disappointment-hedging fund by accounting for their relative downside variance (covariance). Therefore, we refer to this third investor's specific fund as the downside risk-adjusted disappointment hedging fund. Since the weight that the investor assigns to this fund is negative, as shown in equation (19), we argue that the GDA investor would typically reduce (increase) the amount of her position in the disappointment hedging fund in assets where the downside variance is greater (lower) than the standard variance.

The above discussion on the positions and their relative amounts that each of the four funds in equation (19) has on the different assets is straightforward in the case where the matrices Σ_τ and $\Sigma_{\mathcal{D},\tau}$ are diagonal. However, in the general case, standard and downside correlations may be nontrivial in altering the positions and relative amounts that the GDA investor holds in each fund.

Notice from equation (19) that the three funds $w^{\mathbf{DH}}$, $w^{\mathbf{MVA}}$, and $w^{\mathbf{DHA}}$ are all investor-specific, provide downside risk hedging and can therefore be consolidated into a single investor-specific downside risk hedging fund, $w^{\mathbf{GDA}}$, as follows:

$$w = \frac{1}{\tilde{\gamma}_\tau} w^{\mathbf{MV}} + \frac{1}{\tilde{\gamma}_\tau} w^{\mathbf{GDA}} \quad \text{with} \quad w^{\mathbf{GDA}} = \lambda_{\mathcal{D},\tau} w^{\mathbf{DH}} - \alpha_{\mathcal{D},\tau} w^{\mathbf{MVA}} - \lambda_{\mathcal{D},\tau}\alpha_{\mathcal{D},\tau} w^{\mathbf{DHA}}. \tag{22}$$

Each fund in equation (22) can be normalized by the absolute sum of its weights to preserve the nature of the positions (long or short) held in each asset:

$$\bar{w}^{\mathbf{MV}} \equiv \frac{w^{\mathbf{MV}}}{\left|\mathbf{1}^\top w^{\mathbf{MV}}\right|} \quad \text{and} \quad \bar{w}^{\mathbf{GDA}} \equiv \frac{\lambda_{\mathcal{D},\tau} w^{\mathbf{DH}} - \alpha_{\mathcal{D},\tau} w^{\mathbf{MVA}} - \lambda_{\mathcal{D},\tau}\alpha_{\mathcal{D},\tau} w^{\mathbf{DHA}}}{\left|\lambda_{\mathcal{D},\tau}\mathbf{1}^\top w^{\mathbf{DH}} - \alpha_{\mathcal{D},\tau}\mathbf{1}^\top w^{\mathbf{MVA}} - \lambda_{\mathcal{D},\tau}\alpha_{\mathcal{D},\tau}\mathbf{1}^\top w^{\mathbf{DHA}}\right|}. \tag{23}$$

An alternative formulation of the investor's portfolio decomposition in equation (22) leads to a three-fund representation that includes the risk-free asset, the common mean-variance efficient fund $\bar{w}^{\mathbf{MV}}$, and a single investor-specific GDA fund $\bar{w}^{\mathbf{GDA}}$. Under this representation, the optimal portfolio rule in equation (19) can be rewritten as:

$$w = \alpha^{\mathbf{MV}}\bar{w}^{\mathbf{MV}} + \alpha^{\mathbf{GDA}}\bar{w}^{\mathbf{GDA}} \tag{24}$$

where the respective weights assigned to these normalized funds are given by:

$$\alpha^{\mathbf{MV}} \equiv \frac{1}{\tilde{\gamma}}\left|\mathbf{1}^\top w^{\mathbf{MV}}\right| \quad \text{and} \quad \alpha^{\mathbf{GDA}} \equiv \frac{1}{\tilde{\gamma}}\left|\lambda_{\mathcal{D},\tau}\mathbf{1}^\top w^{\mathbf{DH}} - \alpha_{\mathcal{D},\tau}\mathbf{1}^\top w^{\mathbf{MVA}} - \lambda_{\mathcal{D},\tau}\alpha_{\mathcal{D},\tau}\mathbf{1}^\top w^{\mathbf{DHA}}\right|. \tag{25}$$

The weight $\alpha^{\mathbf{GDA}}$ quantifies how much the GDA investor's optimal portfolio deviates from an expected utility (EU) investor with equal endogenous risk aversion. Specifically, relative to such an EU investor, the GDA investor reallocates available cash to modify her risky asset positions by trading in her specific fund. Suppose that a particular structure is imposed on asset returns. In that case, the fund $\bar{w}^{\mathbf{GDA}}$ may become linear in other funds that depend only on the asset return distribution parameters, leading to a standard fund separation strategy. This is particularly the case in Dahlquist et al. (2016). We will further investigate the heterogeneity in the composition of the GDA investor's specific fund across different sets of preference parameters.

4 Empirical assessment

4.1 Data and summary statistics

This section evaluates our theoretical implications using real data on risky assets and the risk-free rate. This evaluation amounts to computing the optimal portfolio choice that maximizes the GDA investor's certainty equivalent for the portfolio gross return over a specified investment horizon. The investment portfolio comprises five asset classes: the risk-free rate proxied by the 3-month Treasury bill secondary market rate sourced from the FRED database (DTB3); equities represented by the S&P500 composite index (S&PCOMP); commodities as represented by the S&P Goldman Sachs commodity index (GSCITOT); bonds represented by the United States 10-Year Government Benchmark Bond index (BMUS10Y); and real estate measured by United States Real Estate Investment Trusts (REITSUS). We use daily data, and our sample ranges from January 2$^{\text{nd}}$, 1989 to October 31$^{\text{st}}$, 2022.

Table 1 presents the summary statistics of excess returns for the four risky asset classes—stocks (**SK**), commodities (**CY**), bonds (**BD**), and real estate (**RE**)—over different investment horizons: one day ($\tau = 1$), five days ($\tau = 5$), ten days ($\tau = 10$) and twenty days ($\tau = 20$). These horizons roughly correspond to daily, weekly, biweekly, and monthly returns, assuming approximately 22 monthly trading days, as is standard in the finance literature (e.g., Corsi; 2009). The multiday return series are constructed using overlapping data, resulting in a sample size of $T - \tau + 1$, where T is the original daily data sample size.

The daily average excess returns, as reported in Table 1, are 0.029% for stock, 0.013% for commodity, 0.013% for bond, and 0.023% for real estate. Their respective standard deviations are 1.123%, 1.348%, 0.446%, and 1.490%, resulting in daily Sharpe ratios ranging from 0.009 for commodity to 0.030 for bond. In particular, all risky assets exhibit excess kurtosis, with real estate showing the highest and bond the lowest. This is further confirmed by the significant differences between the minimums and the fifth quantiles (Q05) and between the maximums and the ninety-fifth quantiles (Q95), highlighting the fat-tailed nature of the return distributions.

Skewness is particularly important in this study on downside risk and its impact on portfolio choice. The daily skewness is negative for all assets (ranging from -0.095 for bonds to -0.498 for the commodity) except for real estate, which exhibits a positive skewness of 0.243. As the horizon lengthens, the skewness of real estate decreases at $\tau = 5$ (0.030) before turning negative at $\tau = 10$ (-0.762) and $\tau = 20$ (-0.660). Similarly, the skewness of the stock and the commodity becomes more negative at longer horizons, reaching -0.794 and -0.495, respectively, at $\tau = 10$, and -0.812 and -0.434 at $\tau = 20$. In contrast, the skewness of the bond returns becomes more positive as the horizon increases, moving from -0.095 on the daily horizon to 0.201 on the monthly horizon. Bond always shows the smallest magnitude among skewness values on different horizons (except for $\tau = 5$).

These findings indicate that standard distributional assumptions (e.g., Dahlquist et al.; 2016), which imply regularity in higher-order moments across horizons (particularly diminishing skewness with increasing horizon), do not hold in the data. This corroborates the findings of Neuberger (2012), who showed that the skewness of the equity index returns increases with the investment horizon, up to a year, with significant economic implications. These results also justify adopting a portfolio estimation approach based on empirical return distributions at each horizon rather than relying on theoretical distributions that fail to capture critical features of aggregated returns.

In terms of correlations, bond maintains negative correlations with stock and commodity across all horizons (ranging from -0.193 at $\tau = 1$ to -0.123 at $\tau = 20$ for stock and -0.164 to -0.265 for commodity). The magnitude of bond-stock's correlation tends to decrease as the horizon increases, whereas it seems to increase for bond-commodity's correlation. Bond also exhibits weaker correlations with real estate, which vary monotonically from -0.104 at $\tau = 1$ to 0.051 at $\tau = 20$. Stocks and real estate remain the most strongly correlated assets, with correlations consistently around 0.66 across horizons.

Table A1 in the external appendix provides the same summary statistics using nonoverlapping data. With nonoverlapping data, the sample size for τ-day returns becomes T/τ, compared to $T - \tau + 1$ for overlapping data. Interestingly, the patterns observed using overlapping data are consistent with those from the nonoverlapping data, but overlapping data offer the advantage of increasing the effective sample size. Since the descriptive statistics using the two approaches are nearly identical in this static context - where serial dependence in the data (affected by overlapping) is irrelevant - the overlapping approach is well justified. Furthermore, a large sample size is critical to capture rare disappointing events under GDA investor preferences, which feature a kink in the utility function. Since the probability of disappointing events may be small in population, a larger sample size helps bring their proportion closer to the true probability by the law of large numbers. The overlapping method improves the precision of our GMM estimates for portfolio weights.

While our empirical analysis emphasizes the economic implications of optimal portfolios under GDA preferences, we do not report standard errors or confidence intervals for portfolio weights or certainty equivalents. This choice reflects a deliberate trade-off between analytical clarity and statistical completeness. Given the large sample size (over 8,000 daily observations) and the use of high-frequency data, the GMM-based portfolio estimates are precise, and sampling variability is minimal in practice. As such, point estimates are enough to illustrate our main economic findings without loss of interpretive rigor. Nonetheless, if needed, statistical inference—such as Wald tests or bootstrap confidence intervals—can be readily implemented for specific quantities of interest.

4.2 Certainty equivalent and premium for downside risk hedging

We begin by analyzing the certainty equivalent of the optimal portfolio, the central financial objective in our framework, and how it varies with the GDA preference parameters and the investment horizon. These results are illustrated in Figure 1A, which plots the certainty equivalent of the optimal portfolio as a function of $\ell \in [0, 3]$ for different investment horizons ($\tau \in \{1, 5, 10, 20\}$) and combinations of preference parameters: risk aversion $\gamma = 3$, and relative disappointment threshold $\kappa \in \{0.95, 0.975, 1, 1.025, 1.05\}$. Certainty equivalent values are annualized using the formula $\mathcal{C}_\tau^{360/\tau} - 1$ and are reported in percentage terms. Generally, the certainty equivalent declines as ℓ increases since a higher disappointment aversion amplifies the investor's effective risk aversion, reducing risk-taking and ultimately lowering the certainty equivalent.

A key observation is that when $\kappa = 1$, regardless of the investment horizon, our numerical results reveal a sharp decrease in the certainty equivalent as the degree of disappointment aversion ℓ increases. Eventually, the certainty equivalent flattens at its minimum value of 2.72%, corresponding to the annualized risk-free rate. This suggests that the investor optimally allocates all her wealth to the risk-free asset and does not participate in the risky asset market. This finding aligns with the results of Ang et al. (2005), who demonstrate that disappointment-averse (DA) investors with $\kappa = 1$ exhibit optimal nonparticipation in the stock market when the degree of disappointment aversion exceeds a critical threshold, i.e., there exists ℓ^* such that the optimal portfolio satisfies $w = 0$ if $\ell \geq \ell^*$. Their proof applies to settings with a single risky asset. To our knowledge, no analytical proof exists for a DA framework with multiple risky assets.

Tédongap and Tinang (2022) extend the theoretical understanding of optimal nonparticipation by proving its occurrence in settings with multiple risky assets. They demonstrate that in a framework where investors minimize portfolio variance subject to linear constraints on expected return and non-normality, non-participation is equivalent to holding a normally distributed portfolio with a non-positive risk premium. This suggests that earlier results on optimal nonparticipation with a single risky asset (Ang et al.; 2005) also extend to settings with multiple risky assets. Dahlquist et al. (2016) further illustrate the optimal nonparticipation in a DA framework but do not provide an analytical characterization. The novelty of our contribution lies in deriving numerical results on the nonparticipation of DA investors without imposing any assumptions on the return distributions, in contrast to Tédongap and Tinang (2022) and Dahlquist et al. (2016), which assume multivariate normal-exponential returns. This result is clearly illustrated in Figure 1A.

Our numerical solutions reveal that ℓ^*, the threshold beyond which DA investors with $\kappa = 1$ invest exclusively in the risk-free asset, depends on the investment horizon. Specifically, we observe that ℓ^* is approximately 0.10, 0.30, 0.45, and 0.70 for the horizons of one day, five days, 10 days, and 20 days, respectively. This implies that for $\kappa = 1$, longer investment horizons require a higher disappointment aversion to trigger complete withdrawal from risky asset markets.[3]

4.2.1 Effect of κ on certainty equivalent

Figure 1A also shows that the certainty equivalent increases for $\kappa < 1$ as κ decreases toward zero, keeping everything else constant. A lower κ reduces the probability of disappointment, encouraging greater risk-taking. For example, the certainty equivalent for $\kappa = 1$ is lower than for $\kappa = 0.975$, which in turn is lower than for $\kappa = 0.95$, and so on, ultimately converging toward the certainty equivalent of the EU investor, that is, the value at $\ell = 0$, where all curves κ intersect in the graph. From equation (9), we observe that as $\kappa < 1$ approaches zero, the set of disappointing events shrinks, reducing the probability of disappointment. As a result, the Euler condition governing the GDA investor's optimal allocation increasingly resembles that of the EU investor, leading to a convergence in their optimal portfolios and certainty equivalents. This observation highlights that all else being equal, the EU investor's allocation is riskier than that of any GDA investor, as it corresponds to a GDA investor with $\kappa < 1$, for whom disappointment never occurs.

[3]Kontosakos et al. (2024) also illustrated in Figure 4 of their study that DA investors with $\kappa = 1$ require higher levels of disappointment aversion to abstain from risky asset markets as the investment horizon increases, implying a decreasing effective risk aversion over longer horizons.

Similarly, for $\kappa > 1$, the certainty equivalent increases as κ increases, keeping all others constant. A higher κ shifts the investor's focus toward more considerable upside gains, encouraging greater risk-taking. For example, the certainty equivalent for $\kappa = 1$ is lower than that for $\kappa = 1.025$, which in turn is lower than that for $\kappa = 1.05$, and so on, ultimately converging to the EU investor's certainty equivalent. From equation (9), as $\kappa > 1$ increases toward infinity, the set of disappointing outcomes expands, driving the probability of disappointment toward one, effectively making disappointment almost sure (since only infinitely high gains prevent disappointment). Consequently, the Euler condition for the GDA investor's optimal allocation increasingly resembles that of the EU investor, leading to similar certainty equivalents. This observation underscores that all else being equal, the EU investor's allocation is riskier than that of any GDA investor, as it corresponds to a GDA investor with $\kappa > 1$, whose decision-making is primarily driven by the pursuit of the most extreme gains.

Interestingly, GDA investors with $\kappa = 0.975$ and $\kappa = 1.025$ exhibit equal but opposite asymmetry in disappointment perception, making them mirror images in their aversion to downside losses versus their desire for upside gains. This is also the case for $\kappa = 0.95$ and $\kappa = 1.05$. Figure 1A finally illustrates that the GDA investor's certainty equivalent converges less rapidly to that of the EU investor with the same standard risk aversion parameter (which has the highest certainty equivalent) as $\kappa < 1$ decreases than when the equal but opposite asymmetry in disappointment perception $\kappa > 1$ increases.

4.2.2 Approximation accuracy

Next, we assess the accuracy of the approximation by comparing the certainty equivalent of the proper optimal portfolio, estimated via GMM using the moment condition (9), to that of the approximated portfolio obtained from the fixed-point solution of equation (19), which is based on the density approximations in (12). Figure 1B illustrates this precision by plotting the ratio of the certainty equivalent derived from the right-hand side of equation (19) to that of the optimal portfolio without approximation. Across all investment horizons, this ratio remains consistently close to 100%, except for the daily horizon and a few cases where ℓ is sufficiently large (i.e., between 2.2 and 3) and $\kappa < 1$. Even in these instances, the approximation remains highly precise, with an accuracy exceeding 98%. Therefore, we can conclude that both the density approximations in Equation (12) and the portfolio decomposition in Equation (19) are highly accurate. All certainty equivalent values reported in Table 2 further illustrate and confirm this conclusion.

4.2.3 Certainty equivalent cost of ignoring downside risk aversion

Finally, we evaluate the accuracy of approximating the GDA investor's optimal certainty equivalent using that of the standard mean-variance (MV) portfolio, $\frac{1}{\tilde{\gamma}^e_\tau} w^{\mathbf{MV}}$, where $\tilde{\gamma}^e_\tau$ is the effective risk aversion (see Section 2.2, equation (18)). The difference in certainty equivalents between the two portfolios quantifies the cost of ignoring disappointment aversion, representing the additional premium the GDA investor demands to compensate for downside risk aversion. It highlights the extent to which the MV portfolio is suboptimal due to its inability to account for asymmetric risk perception, particularly concerning downside losses.

Figure 1B shows the ratio of MV investor's certainty equivalent to that of the GDA investor, indicating how well the MV portfolio captures the preferences of a disappointment-averse investor. Although the MV framework does not explicitly model disappointment aversion, it aligns partially with the GDA investor's utility through standard risk-return trade-offs. The remaining gap quantifies the shortfall resulting from the MV investor's inability to hedge the risk of disappointment.

Regardless of κ, at $\ell = 0$, the GDA preferences reduce to the EU preferences, thus almost MV, making the certainty equivalents identical and the ratio 100%. For $\kappa = 1$, the ratio follows a U-shape for $0 \leq \ell \leq \ell^*$. At ℓ^*, where the DA investor abstains from risky asset markets, both certainty equivalents equal the risk-free rate, maintaining a 100% ratio.[4] Given that the ratio is 100% at both $\ell = 0$ and ℓ^*, Rolle's theorem implies a local minimum. Numerically, we observe this minimum across all investment horizons. Although it decreases as the horizon increases, it remains at 93% at the 20-day horizon, meaning up to 7% of the certainty equivalent remains unaccounted for. This highlights that while the MV portfolio captures much of the DA investor's preferences, it fails to hedge downside risk effectively, particularly in relatively short horizons considered here, where the missing 7% represents a significant cost of ignoring disappointment aversion.

The MV portfolio is even less effective for GDA investors with $\kappa \neq 1$ as ℓ increases. At the 20-day horizon, for $\ell \geq 1.5$, the ratio drops below 90% for $\kappa > 1$ and below 86% for

[4]This also holds for $\ell > \ell^*$.

$\kappa < 1$. The remaining 14% or more in the latter case reflects the premium for downside risk hedging—the cost the GDA investor is willing to pay (or the loss she incurs) due to the MV portfolio's inability to address her asymmetric perception of risk.

4.3 Optimal portfolio weights

We now discuss the optimal portfolio weights and examine how these weights vary with the GDA preference parameters and the investment horizon. Actual values are estimated via GMM using the moment condition (9), and approximated values are solved through the non-linear fixed-point equation (19), based on the density approximations in (12). Finally, we compare the certainty equivalent of the optimal portfolio of the GDA investor with that of the optimal portfolio of the MV investor (calibrated to the same endogenous risk aversion, $\tilde{\gamma}_\tau$).

Table 2 presents the estimated optimal portfolio weights and corresponding certainty equivalent values for different investment horizons ($\tau \in \{1, 5, 10, 20\}$) and combinations of GDA preference parameters: risk aversion $\gamma = 3$, disappointment aversion $\ell \in \{0, 1, 2, 3\}$ and relative disappointment threshold $\kappa \in \{0.95, 0.975, 1, 1.025, 1.05\}$. All portfolio weights and certainty equivalent values are reported in percentage terms.

We begin by analyzing the results for the daily investment horizon ($\tau = 1$), which serve as a reference since they are qualitatively representative of the findings across longer horizons, all else equal. We then explore how the investment horizon affects optimal portfolio allocations and the certainty equivalents under GDA preferences.

4.3.1 Daily investment horizon

We first focus on the daily investment horizon and analyze the expected utility preferences ($\ell = 0$). The optimal weights obtained without the approximation (column (N)) suggest that the EU investor borrows heavily in cash (-288.57% weight in the optimal portfolio) to invest substantially in bonds (275.21%) and stocks (100.08%). For the EU investor, only the first term on the right-hand side of equation (19) is relevant, namely the MV term, as all remaining terms are null. Acting as a standard MV investor, as discussed in Section 2.2, the EU investor seeks to maximize the Sharpe ratio. Consequently, she maintains substantial long positions in the assets with the highest Sharpe ratios in our sample (**BD** and **SK**).

Furthermore, given the negative correlation between **BD** and **SK**, long positions in both assets diversify risk and further enhance the Sharpe ratio. To further diversify, the EU investor moderately buys commodities (23.41%) and short-sells real estate (−10.13%). The short position in real estate is compatible with its strong positive correlation with stocks, which meets diversification requirements.

Interestingly, we observe that the approximated analytical solution (column (Y)), the left-hand side of equation (19), yields portfolio weights very close to the exact numerical solution. Furthermore, the certainty equivalent achieved is identical, equal to 15.66%. This confirms the accuracy of the approximation.

We now look at the optimal allocation of the investor with disappointment aversion ($\kappa = 1$). All values of ℓ reported in the table are higher than the threshold $\ell^* \approx 0.10$, from which there is no participation in risky asset markets when the investment horizon is one day. Therefore, the investor prefers safety, optimally holding no risky assets, thus investing all her wealth in cash. The annualized certainty equivalent in this case corresponds to the risk-free rate of 2.72%.

We observe in Figure 1A that the certainty equivalent for the EU investor is higher than for any GDA investor, all else equal. This reflects the greater effective risk tolerance of the EU investor. Figure 2A illustrates this effective risk tolerance, showing that all κ curves initially intersect at $\ell = 0$, corresponding to the expected utility with a risk tolerance of 1/3, before decreasing for GDA investors as ℓ increases. This reduction in risk tolerance must be reflected in portfolio allocations. Table 2 confirms that the EU investor holds the most significant long positions in stock, commodity, and bond compared to any GDA investor.

Next, we examine cases where the disappointment threshold deviates from the certainty equivalent, i.e., κ is lower than or higher than one. Figure 1A shows that the certainty equivalent for $\kappa = 0.975$ is lower than for $\kappa = 0.95$, indicating that the former GDA investor is more risk-averse than the latter. This increased risk aversion is also evident in Figure 2A, where the effective risk tolerance curve for $\kappa = 0.975$ is below $\kappa = 0.95$ as ℓ increases.

This pattern is reflected in the portfolio allocations in Table 2. Consider the case with $\ell = 2$. For a GDA investor with $\kappa = 0.975$ - where disappointment sets in if the result is at least 2.5% worse than the certainty equivalent - the portfolio weights obtained without approximation are 45.65% in stock, 6.38% in commodity, 123.99% in bond and −10.13% in real estate, resulting in an overall long position of 165.89%. In contrast, for a GDA investor with $\kappa = 0.95$ - where disappointment occurs if the outcome is at least 5% worse than the certainty equivalent - these allocations increase in magnitude to 74.94% in stock, 16.25% in commodity, 202.99% in bonds and -17.47% in real estate, leading to a total long position of 276.71%.

Similarly, Figure 1A shows that the certainty equivalent of $\kappa = 1.025$ is lower than for $\kappa = 1.05$, indicating that the former GDA investor is more risk averse than the latter. This trend is further confirmed by the effective risk tolerance curves in Figure 2A, where the curve for $\kappa = 1.025$ is consistently lower than that for $\kappa = 1.05$. The impact of this difference in risk tolerance is also reflected in the portfolio allocations presented in Table 2.

Focusing again on the case with $\ell = 2$, a GDA investor with $\kappa = 1.025$ - where disappointment occurs unless the outcome is at least 2.5% better than the equivalent certainty - holds an overall risky asset position of 244.33%. In contrast, for a GDA investor with $\kappa = 1.05$—where disappointment occurs unless the outcome is at least 5% better than the equivalent certainty—this position increases to 343.40%, reflecting the investor's higher risk tolerance.

4.3.2 MV versus GDA allocation and hedging component

We now examine how the allocation $\frac{1}{\tilde{\gamma}_\tau} w^{\mathbf{MV}}$—the first term on the right-hand side of Equation (22)—compares with the optimal allocation of the GDA investor. This term represents the optimal allocation of an MV investor with risk aversion $\tilde{\gamma}_\tau$, equivalent to the endogenous risk aversion. Naturally, this allocation is suboptimal for the GDA investor, leading to a lower certainty equivalent. The difference between the two allocations, given by $\frac{1}{\tilde{\gamma}_\tau} w^{\mathbf{GDA}}$, constitutes the GDA investor's hedging component (see equation (22)).

The allocation $\frac{1}{\tilde{\gamma}_\tau} w^{\mathbf{MV}}$ also corresponds to $\alpha^{\mathbf{MV}} \bar{w}^{\mathbf{MV}}$, while the allocation $\frac{1}{\tilde{\gamma}_\tau} w^{\mathbf{GDA}}$ corresponds to $\alpha^{\mathbf{GDA}} \bar{w}^{\mathbf{GDA}}$ (see Equations (23) and (24)). Table 3 presents the normalized **MV** and **GDA** funds alongside their respective weights in the optimal portfolio, as well as the relative downside variance of the assets $\frac{\sigma^2_{i\mathcal{D},\tau} - \sigma^2_{i,t}}{\sigma^2_{i,t}}$, all expressed in percentage terms.

Across all scenarios in Table 3, the sum of weights in the normalized **MV** fund is consistently 100%, while it is -100% for the normalized **GDA** fund, with both funds carrying positive weights in the optimal portfolio. This implies that the GDA investor takes a long position in the common normalized **MV** fund while shorting through her specific normalized **GDA** fund. Consequently, the net exposure to risky assets is given by $\alpha^{\mathbf{MV}} - \alpha^{\mathbf{GDA}}$. This position remains long in all scenarios.

By investing in her specific **GDA** fund, the GDA investor effectively reduces risk exposure from the typical **MV** fund, reinforcing the interpretation of the **GDA** fund as a hedging mechanism. Longing the common **MV** fund (the first term on the right-hand side of Equation (24)) appears excessively risky to the GDA investor, who must therefore short her specific **GDA** fund (the second term in Equation (24)) to achieve an optimal portfolio with a risk level aligned with her preferences.

This adjustment quantifies how incorporating downside risk hedging mitigates risk exposure, ultimately transforming a riskier allocation into an optimal portfolio tailored to the GDA investor's preferences. It highlights the crucial role of endogenous downside risk hedging in portfolio selection under GDA preferences, illustrating how deviations from traditional mean-variance strategies address asymmetries in risk perception.

The certainty equivalent of exclusively longing the common **MV** fund is reported in Table 2. For $\tau = 1$, consider the case where $\ell = 2$ and examine a GDA investor with $\kappa = 0.975$. If she invests exclusively in the common **MV** fund, her certainty equivalent is -19.07%, indicating that this strategy is perilous from her point of view. As a result, she would effectively pay an implicit insurance premium to avoid such an allocation since her optimal portfolio achieves a certainty equivalent of 9.97%.

In contrast, a GDA investor with $\kappa = 0.95$ (respectively, $\kappa = 1.025$ and $\kappa = 1.05$) values the common **MV** fund more favorably, as she is willing to accept a certainty equivalent of 8.89% (respectively, 6.42% and 14.40%) for this strategy. However, this allocation remains suboptimal, requiring an adjustment through her specific hedging fund to achieve the certainty equivalent of 13.31% (respectively, 12.07% and 14.76%) associated with her optimal portfolio.

4.3.3 Fund weights and implications for effective risk aversion

Table 3 shows that the weight assigned to the **MV** fund remains relatively stable in different combinations of preference parameters. For example, at $\tau = 1$, $\alpha^{\mathbf{MV}}$ varies narrowly between 386.87% and 390.18%. This consistency reflects the limited variation in the endogenous risk aversion parameter $\tilde{\gamma}_\tau$ between scenarios (see equation (25)), as further illustrated in Figure

2B, which plots the corresponding endogenous risk tolerance $1/\tilde{\gamma}_\tau$.

In contrast, the weight assigned to the **GDA** fund, $\alpha^{\mathbf{GDA}}$, varies significantly, especially concerning the disappointment threshold parameter κ. Specifically, $\alpha^{\mathbf{GDA}}$ is largest when κ is near one and decreases as κ moves away from one in either direction. Focusing on the case $\ell = 2$, $\alpha^{\mathbf{GDA}}$ equals 227.46% for $\kappa = 0.975$ and 145.10% for $\kappa = 1.025$, but drops to 116.09% for $\kappa = 0.95$ and 44.15% for $\kappa = 1.05$.

An increase in $\alpha^{\mathbf{GDA}}$ relative to $\alpha^{\mathbf{MV}}$ reduces the net exposure to risky assets, given by $\alpha^{\mathbf{MV}} - \alpha^{\mathbf{GDA}}$, thus signaling a higher effective risk aversion. This trend is further confirmed in Panel C of Figure 2, which plots the ratio $\alpha^{\mathbf{GDA}}/\alpha^{\mathbf{MV}}$ between preference configurations. The curves for $\kappa = 0.975$ and $\kappa = 1.025$ consistently lie above those for $\kappa = 0.95$ and $\kappa = 1.05$, respectively, confirming that GDA investors with κ closer to one exhibit lower risk tolerance than those with more extreme values of κ. The ratio is highest for $\kappa = 1$, all else equal. Moreover, $\alpha^{\mathbf{GDA}}/\alpha^{\mathbf{MV}}$ increases with ℓ, reinforcing its usefulness as a proxy for effective risk aversion.

As discussed earlier, GDA investors with $\kappa = 0.975$ (resp. $\kappa = 0.95$) and $\kappa = 1.025$ (resp. $\kappa = 1.05$) exhibit equal but opposite asymmetry in the perception of disappointment. Examining their reliance on the **GDA** fund to achieve optimality reveals that investors with $\kappa < 1$, due to their aversion to downside losses, reduce their exposure to risky assets more substantially than their counterparts with $\kappa > 1$, who are motivated by a desire for upside gains. This asymmetry is reflected in Figure 2C, where the $\kappa < 1$ curves are consistently above their counterparts with $\kappa > 1$.

Further insight into this differential behavior is provided by examining the relative downside variance (RDV) values of the risky assets reported in Table 3. The RDV values indicate that optimal portfolios for GDA investors with $\kappa < 1$ typically exhibit significant and positive RDV, whereas those for $\kappa > 1$ exhibit negative RDV values that are smaller in magnitude. These findings support a long-standing observation in economics: aversion to downside losses exerts a more substantial influence on behavior than the desire to capture upside gains.

4.3.4 Effect of investment horizon τ

Having discussed the daily investment horizon results as a benchmark, we now explore how optimal portfolio weights and overall risk tolerance evolve as the investment horizon increases.

Starting with the EU investor in Table 2, some clear patterns emerge as τ increases. In particular, the long position in stock increases from 100.08% at $\tau = 1$ to 129.01% at $\tau = 20$, while the short position in real estate increases from -10.13% to -40.39%. In contrast, allocations to other assets, particularly bonds, which consistently receive the highest weight, do not follow a monotonic trend.

This irregularity results in relatively stable certainty equivalents, ranging from 15.41% at $\tau = 5$ to 16.10% at $\tau = 20$, a spread of only 0.68%. This is the narrowest variation observed across all preference configurations, indicating that the EU investor's overall exposure to risky assets remains constant across investment horizons. This pattern is confirmed by the stability of the $\alpha^{\mathbf{MV}}$ values in Table 3.

In contrast, for all GDA investors in Table 2, allocations to stocks and bonds—the two dominant assets—decline monotonically as the investment horizon increases, driving a similar downward trend in the certainty equivalent. For instance, with $\ell = 2$ and $\kappa = 0.975$, the stock (bond) allocation falls from 45.65% (123.99%) at $\tau = 1$, yielding a certainty equivalent of 9.97%, to 27.29% (55.30%) at $\tau = 20$, where the certainty equivalent drops to 6.17%. This corresponds to a certainty equivalent spread of -3.80%.

At first glance, this decline in risk-taking with the horizon may seem counterintuitive. The finance literature typically asserts that longer horizons promote higher risk tolerance. This view stems from standard expected utility models with mean-reverting returns, where the variance of cumulative returns increases less rapidly than the mean, making long-horizon investments more attractive (see, e.g., Cochrane; 2007 and references therein). It also aligns with widespread advice from professionals (e.g., Bogle; 1999; Malkiel; 1996). However, our findings show that this logic breaks down for investors with generalized disappointment aversion preferences. For these agents, longer horizons increase the psychological salience of disappointment, leading to more conservative portfolio allocations, even when more time is available to absorb risk.

To understand this mechanism, consider GDA investors with $\kappa < 1$, who focus on minimizing downside losses. The top two panels in Figure 3A plot the probability of disappointment $\pi_{\mathcal{D},\tau}$ as a function of the penalty coefficient $\ell \in [0, 3]$ for investment horizons $\tau \in \{1, 5, 10, 20\}$. As ℓ increases, investors assign more weight to disappointment and reduce its likelihood, causing $\pi_{\mathcal{D},\tau}$ to decline. However, as τ increases, $\pi_{\mathcal{D},\tau}$ increases for fixed ℓ since the probability of experiencing a downside loss increases with the horizon. This is evident as the curve for $\tau = 20$ lies above those for shorter horizons.

A higher probability of disappointment implies a greater effective risk aversion. Consequently, the two top panels of Figure 3C show that risk tolerance decreases with the horizon when $\kappa < 1$, with the curve $\tau = 20$ consistently below the others.

Now consider GDA investors with $\kappa > 1$, who focus on achieving upside gains. The two bottom panels in Figure 3B display the expected excess return conditional on no disappointment as a function of ℓ for various horizons. These investors are motivated by the size of the potential gains. However, due to the effects of averaging and the reduced impact of rare positive shocks on longer horizons, the expected upside gain decreases as τ increases. Therefore, achieving a considerable gain in one day requires more risk-taking than achieving the same gain over twenty days, making short-horizon GDA investors with $\kappa > 1$ more risk-tolerant than their long-horizon counterparts. This trend is confirmed in the corresponding panels of Figure 3C.

These findings resonate with the well-known Allais paradox, which distinguishes between behavior under a downward and upward focus. When disappointment arises from losses ($\kappa < 1$), investors prioritize minimizing the probability of unfavorable outcomes. When disappointment comes from insufficient gains ($\kappa > 1$), they focus on maximizing the size of positive outcomes. In both cases, longer horizons either raise the likelihood of disappointment or reduce the perceived size of the potential upside, leading to more conservative portfolio behavior.

4.3.5 Additional horizon effects

To complete our analysis, we turn to horizon effects under the canonical disappointment aversion (DA) model of Gul (1991), where the disappointment threshold coincides with the certainty equivalent ($\kappa = 1$). Previous sections focused exclusively on the generalized disappointment aversion (GDA) framework of Routledge and Zin (2010), which allows the threshold to deviate from the certainty equivalent and predicts declining risk-taking with horizon—opposite to standard theory. Panel A1 of Figure 4 reveals that, unlike GDA, DA preferences generate a mildly increasing risk tolerance with the investment horizon. When disappointment aversion is low, the certainty equivalent increases monotonically with the horizon. Thus, the decline in risk-taking at short horizons is a specific implication of GDA, not a general feature of disappointment-based preferences.

We extend our analysis beyond short horizons to examine behavior over several months, as typically studied in the portfolio choice literature. While our exact solution (Equation (9)) remains valid at any horizon, the fixed-point approximation (Equation (19)) depends on the quality of the density approximations in Equation (12), which may deteriorate with the horizon. To evaluate this, we fix preference parameters at $\gamma = 3$ and $\ell = 2$ and study how key quantities evolve with the investment horizon τ. Panel A2 of Figure 4 plots the approximate to exact certainty equivalent ratio as a function of τ, showing that the approximation remains accurate—within 2% error—for horizons up to 14 months. The associated mean-variance structure is thus reliable at sufficiently short horizons, which remains the empirical focus of this paper.

Panel A3 of Figure 4 plots the certainty equivalent across horizons, revealing a U-shaped risk profile: risk-taking initially declines with the horizon but begins to rise again once the horizon becomes sufficiently long. We have already provided a behavioral explanation for the short-horizon decline: The psychological characteristics embedded in GDA preferences dominate, leading to more cautious portfolio allocations at these horizons. However, the statistical properties of asset returns become more favorable as the horizon extends. The distributional improvements eventually outweigh behavioral frictions, restoring the classical risk-horizon relationship.

Panels B1 to B3 of Figure 4 illustrate this shift. The annualized Sharpe ratios of bond and real estate rise sharply with the horizon, while the Sharpe ratio of stock remains broadly stable. Skewness becomes less negative for stock and real estate and increasingly positive for commodity. Meanwhile, excess kurtosis—initially significant at short horizons—declines steadily across all risky assets, approaching zero over longer horizons. Together, these changes enhance the attractiveness of risk-taking over more extended periods, aligning with conventional portfolio theory.

Our contribution lies in uncovering the short-horizon implications of GDA preferences. This setting is largely overlooked in the literature despite its importance in practice, particularly for risk management, short-dated products, and high-frequency investment mandates. While long horizons eventually recover standard predictions, our findings highlight how behavioral frictions can fundamentally reshape risk-taking behavior at very short horizons.

5 Conclusion

This paper develops a tractable, model-free framework for portfolio selection under generalized disappointment aversion and provides an empirically grounded analysis using daily data across major asset classes. Despite kinked and asymmetric preferences, we demonstrate that optimal portfolios can be represented endogenously in terms of mean and variance with preference-adjusted moments and effective risk aversion. This structure enables a closed-form decomposition into standard and downside-specific components, providing new insights into how psychological frictions influence asset demand.

Empirically, we document a robust and novel horizon-risk pattern: under GDA preferences, investors reduce risk exposure as the investment horizon extends from one day to approximately one month. This behavior contradicts classical predictions and reflects the heightened salience of downside risk at short horizons. As the horizon expands, statistical improvements in return distributions gradually offset behavioral frictions, resulting in a U-shaped risk profile.

While our primary focus is on portfolio allocation, the endogenous mean-variance structure also yields a behavioral performance measure—the *endogenous Sharpe ratio*, which adjusts for investor-specific downside risk. This metric, explored in the online appendix, naturally connects to existing measures such as the Sortino and Conditional Sharpe ratios, offering a preference-sensitive lens on portfolio efficiency. Future work may extend this framework to fund evaluation, benchmarking, and risk reporting under nonstandard preferences.

APPENDIX

A First-order conditions for utility maximization

The goal of the investor is to choose the portfolio strategy w that maximizes the certainty equivalent $\mathcal{C}_\tau = \mathcal{C}\left(R_{w,\tau}\right)$. Taking the derivative of both sides of equation (8) with respect to w, we obtain

$$\eta U'\left(\mathcal{C}_\tau\right)\frac{\partial \mathcal{C}_\tau}{\partial w} = \mathbb{E}\left[U'\left(R_{w,\tau}\right)\frac{\partial R_{w,\tau}}{\partial w}\right] - \ell \mathbb{E}\left[\left(U'\left(\kappa \mathcal{C}_\tau\right)\kappa \frac{\partial \mathcal{C}_\tau}{\partial w} - U'\left(R_{w,\tau}\right)\frac{\partial R_{w,\tau}}{\partial w}\right)\mathbb{I}\left(R_{w,\tau} < \kappa \mathcal{C}_\tau\right)\right],$$

where U' is the first-order derivative of U given by $U'(x) = x^{-\gamma}$, and where $\frac{\partial R_{w,\tau}}{\partial w} = R_\tau - \mathbf{1}R_{0,\tau}$. At the optimum we must have $\frac{\partial \mathcal{C}_\tau}{\partial w} = 0$. Substituting everything in the above equation leads to the following equation for the optimal portfolio

$$\mathbb{E}\left[R_{w,\tau}^{-\gamma}\left(1 + \ell \mathbb{I}\left(R_{w,\tau} < \kappa \mathcal{C}_\tau\right)\right)\left(R_\tau - \mathbf{1}R_{0,\tau}\right)\right] = 0 \tag{A1}$$

where $\mathcal{C}_\tau$ is implicitly given by equation (8).

B Effective mean-variance interpretation.

We want to find the effective risk aversion, $\tilde{\gamma}_\tau^e$ such that :

$$\begin{aligned}
\mu_{w,\tau} - \frac{\tilde{\gamma}_\tau^e}{2}\sigma_{w,\tau}^2 &\equiv \tilde{\mu}_{w,\tau} - \frac{\tilde{\gamma}_\tau}{2}\tilde{\sigma}_{w,\tau}^2 \\
&= \frac{1}{\tilde{\gamma}_\tau}\tilde{\mu}_\tau^\top \tilde{\Sigma}_\tau^{-1}\tilde{\mu}_\tau - \frac{\tilde{\gamma}_\tau}{2}\frac{1}{\tilde{\gamma}_\tau}\tilde{\mu}_\tau^\top \tilde{\Sigma}_\tau^{-1}\tilde{\Sigma}_\tau \frac{1}{\tilde{\gamma}_\tau}\tilde{\Sigma}_\tau^{-1}\tilde{\mu}_\tau \\
&= \frac{1}{\tilde{\gamma}_\tau}\tilde{\mu}_\tau^\top \tilde{\Sigma}_\tau^{-1}\tilde{\mu}_\tau - \frac{1}{2\tilde{\gamma}_\tau}\tilde{\mu}_\tau^\top \tilde{\Sigma}_\tau^{-1}\tilde{\mu}_\tau \\
&= \frac{1}{2\tilde{\gamma}_\tau}\tilde{\mu}_\tau^\top \tilde{\Sigma}_\tau^{-1}\tilde{\mu}_\tau
\end{aligned} \tag{B1}$$

The left hand of equation (B1) can also be expressed as follows:

$$\begin{aligned} \mu_{w,\tau} - \frac{\tilde{\gamma}_{\tau}^{e}}{2}\sigma_{w,\tau}^{2} &= \frac{1}{\tilde{\gamma}_{\tau}}\tilde{\mu}_{\tau}^{\top}\tilde{\Sigma}_{\tau}^{-1}\mu_{\tau} - \frac{\tilde{\gamma}_{\tau}^{e}}{2}\frac{1}{\tilde{\gamma}_{\tau}}\tilde{\mu}_{\tau}^{\top}\tilde{\Sigma}_{\tau}^{-1}\Sigma_{\tau}\frac{1}{\tilde{\gamma}_{\tau}}\tilde{\Sigma}_{\tau}^{-1}\tilde{\mu}_{\tau} \\ &= \frac{1}{\tilde{\gamma}_{\tau}}\left(\tilde{\mu}_{\tau}^{\top}\tilde{\Sigma}_{\tau}^{-1}\mu_{\tau} - \frac{\tilde{\gamma}_{\tau}^{e}}{2\tilde{\gamma}_{\tau}}\tilde{\mu}_{\tau}^{\top}\tilde{\Sigma}_{\tau}^{-1}\Sigma_{\tau}\tilde{\Sigma}_{\tau}^{-1}\tilde{\mu}_{\tau}\right) \end{aligned} \tag{B2}$$

Equalizing (B1) and (B2), we obtain:

$$\frac{\tilde{\gamma}_{\tau}^{e}}{2\tilde{\gamma}_{\tau}}\tilde{\mu}_{\tau}^{\top}\tilde{\Sigma}_{\tau}^{-1}\Sigma_{\tau}\tilde{\Sigma}_{\tau}^{-1}\tilde{\mu}_{\tau} = \left(\tilde{\mu}_{\tau}^{\top}\tilde{\Sigma}_{\tau}^{-1}\mu_{\tau} - \frac{1}{2}\tilde{\mu}_{\tau}^{\top}\tilde{\Sigma}_{\tau}^{-1}\tilde{\mu}_{\tau}\right) \tag{B3}$$

Therefore,

$$\tilde{\gamma}_{\tau}^{e} = \tilde{\gamma}_{\tau}\frac{\left(2\mu_{\tau} - \tilde{\mu}_{\tau}\right)^{\top}\tilde{\Sigma}_{\tau}^{-1}\tilde{\mu}_{\tau}}{\tilde{\mu}_{\tau}^{\top}\tilde{\Sigma}_{\tau}^{-1}\Sigma_{\tau}\tilde{\Sigma}_{\tau}^{-1}\tilde{\mu}_{\tau}} \tag{B4}$$

C Approximation and portfolio characterization

Recalling $R_{\tau}^{e} \equiv R_{\tau} - \mathbf{1}R_{0,\tau}$ as the vector of excess returns on risky assets, and $\mathcal{D}_{\tau} \equiv \left\{R_{w,\tau} < \kappa\mathcal{C}_{\tau}\right\}$ as the investor's disappointing event, the above first-order condition for optimality, (A1), may also be written

$$\begin{aligned} &\mathbb{E}\left[R_{w,\tau}^{-\gamma}R_{\tau}^{e}\right] + \ell\mathbb{E}\left[R_{w,\tau}^{-\gamma}R_{\tau}^{e}\mathbb{I}\left(\mathcal{D}_{\tau}\right)\right] = 0 \\ &\mathbb{E}\left[R_{w,\tau}^{-\gamma}\right]\frac{\mathbb{E}\left[R_{w,\tau}^{-\gamma}R_{\tau}^{e}\right]}{\mathbb{E}\left[R_{w,\tau}^{-\gamma}\right]} + \ell\mathbb{E}\left[R_{w,\tau}^{-\gamma}\mathbb{I}\left(\mathcal{D}_{\tau}\right)\right]\frac{\mathbb{E}\left[R_{w,\tau}^{-\gamma}R_{\tau}^{e}\mathbb{I}\left(\mathcal{D}_{\tau}\right)\right]}{\mathbb{E}\left[R_{w,\tau}^{-\gamma}\mathbb{I}\left(\mathcal{D}_{\tau}\right)\right]} = 0 \\ &\frac{\mathbb{E}\left[R_{w,\tau}^{-\gamma}R_{\tau}^{e}\right]}{\mathbb{E}\left[R_{w,\tau}^{-\gamma}\right]} + \ell\frac{\mathbb{E}\left[R_{w,\tau}^{-\gamma}\mathbb{I}\left(\mathcal{D}_{\tau}\right)\right]}{\mathbb{E}\left[R_{w,\tau}^{-\gamma}\right]}\frac{\mathbb{E}\left[R_{w,\tau}^{-\gamma}R_{\tau}^{e}\mathbb{I}\left(\mathcal{D}_{\tau}\right)\right]}{\mathbb{E}\left[R_{w,\tau}^{-\gamma}\mathbb{I}\left(\mathcal{D}_{\tau}\right)\right]} = 0 \end{aligned}$$

or equivalently

$$\mathbb{E}\left[\frac{R_{w,\tau}^{-\gamma}}{\mathbb{E}\left[R_{w,\tau}^{-\gamma}\right]}R_{\tau}^{e}\right] + \ell\pi_{\mathcal{D},\tau}^{*}\mathbb{E}\left[\frac{R_{w,\tau}^{-\gamma}}{\mathbb{E}\left[R_{w,\tau}^{-\gamma} \mid \mathcal{D}_{\tau}\right]}R_{\tau}^{e} \mid \mathcal{D}_{\tau}\right] = 0 \tag{C1}$$

with

$$\pi^*_{\mathcal{D},\tau} \equiv \frac{\mathbb{E}\left[R_{w,\tau}^{-\gamma}\mathbb{I}\left(\mathcal{D}_\tau\right)\right]}{\mathbb{E}\left[R_{w,\tau}^{-\gamma}\right]} = \mathbb{E}\left[\frac{R_{w,\tau}^{-\gamma}}{\mathbb{E}\left[R_{w,\tau}^{-\gamma}\right]}\mathbb{I}\left(\mathcal{D}_\tau\right)\right] = \mathbb{E}^*\left[\mathbb{I}\left(\mathcal{D}_\tau\right)\right] = \text{prob}^*\left(\mathcal{D}_\tau\right), \tag{C2}$$

where $\mathbb{E}^*[\cdot]$ denotes the beliefs-implied expectation operator induced by the change-of-measure $\frac{R_{w,\tau}^{-\gamma}}{\mathbb{E}\left[R_{w,\tau}^{-\gamma}\right]}$, and $\text{prob}^*(\cdot)$ denotes the associated beliefs-implied probability operator.

From equation (C1), we consider the following density approximations:

$$\frac{R_{w,\tau}^{-\gamma}}{\mathbb{E}\left[R_{w,\tau}^{-\gamma}\right]} \approx 1 - \frac{\sigma\left[R_{w,\tau}^{-\gamma}\right]}{\mathbb{E}\left[R_{w,\tau}^{-\gamma}\right]}\frac{R_{w,\tau}^{e} - \mathbb{E}\left[R_{w,\tau}^{e}\right]}{\sigma\left[R_{w,\tau}^{e}\right]} = 1 - \gamma_\tau\left(R_{w,\tau}^{e} - \mathbb{E}\left[R_{w,\tau}^{e}\right]\right)$$

$$\frac{R_{w,\tau}^{-\gamma}}{\mathbb{E}\left[R_{w,\tau}^{-\gamma} \mid \mathcal{D}_\tau\right]} \approx 1 - \frac{\sigma\left[R_{w,\tau}^{-\gamma} \mid \mathcal{D}_\tau\right]}{\mathbb{E}\left[R_{w,\tau}^{-\gamma} \mid \mathcal{D}_\tau\right]}\frac{R_{w,\tau}^{e} - \mathbb{E}\left[R_{w,\tau}^{e} \mid \mathcal{D}_\tau\right]}{\sigma\left[R_{w,\tau}^{e} \mid \mathcal{D}_\tau\right]} = 1 - \gamma_{\mathcal{D},\tau}\left(R_{w,\tau}^{e} - \mathbb{E}\left[R_{w,\tau}^{e} \mid \mathcal{D}_\tau\right]\right),$$

where

$$\gamma_\tau = \frac{\sigma\left[R_{w,\tau}^{-\gamma}\right]}{\mathbb{E}\left[R_{w,\tau}^{-\gamma}\right]}\bigg/\sigma\left[R_{w,\tau}^{e}\right] \quad \text{and} \quad \gamma_{\mathcal{D},\tau} = \frac{\sigma\left[R_{w,\tau}^{-\gamma} \mid \mathcal{D}_\tau\right]}{\mathbb{E}\left[R_{w,\tau}^{-\gamma} \mid \mathcal{D}_\tau\right]}\bigg/\sigma\left[R_{w,\tau}^{e} \mid \mathcal{D}_\tau\right]. \tag{C3}$$

Substituting out the density approximations into equation (C1) and rearranging, we obtain

$$\begin{aligned}
&\left(\mathbb{E}\left[R_\tau^e\right] - \gamma_\tau \operatorname{cov}\left(R_{w,\tau}^e, R_\tau^e\right)\right) + \ell\pi^*_{\mathcal{D},\tau}\left(\mathbb{E}\left[R_\tau^e \mid \mathcal{D}_\tau\right] - \gamma_{\mathcal{D},\tau}\operatorname{cov}\left(R_{w,\tau}^e, R_\tau^e \mid \mathcal{D}_\tau\right)\right) = 0\\
&\left(\mu_\tau - \gamma_\tau\Sigma_\tau w\right) + \ell\pi^*_{\mathcal{D},\tau}\left(\mu_{\mathcal{D},\tau} - \gamma_{\mathcal{D},\tau}\Sigma_{\mathcal{D},\tau} w\right) = 0\\
&w = \left(\gamma_\tau\Sigma_\tau + \ell\pi^*_{\mathcal{D},\tau}\gamma_{\mathcal{D},\tau}\Sigma_{\mathcal{D},\tau}\right)^{-1}\left(\mu_\tau + \ell\pi^*_{\mathcal{D},\tau}\mu_{\mathcal{D},\tau}\right)\\
&w = \frac{1+\ell\pi^*_{\mathcal{D},\tau}}{\gamma_\tau + \ell\pi^*_{\mathcal{D},\tau}\gamma_{\mathcal{D},\tau}}\left(\frac{\gamma_\tau\Sigma_\tau}{\gamma_\tau + \ell\pi^*_{\mathcal{D},\tau}\gamma_{\mathcal{D},\tau}} + \frac{\ell\pi^*_{\mathcal{D},\tau}\gamma_{\mathcal{D},\tau}\Sigma_{\mathcal{D},\tau}}{\gamma_\tau + \ell\pi^*_{\mathcal{D},\tau}\gamma_{\mathcal{D},\tau}}\right)^{-1}\left(\frac{\mu_\tau}{1+\ell\pi^*_{\mathcal{D},\tau}} + \frac{\ell\pi^*_{\mathcal{D},\tau}\mu_{\mathcal{D},\tau}}{1+\ell\pi^*_{\mathcal{D},\tau}}\right),
\end{aligned} \tag{C4}$$

where $\mu_\tau = \mathbb{E}\left[R^e_\tau\right]$ and $\Sigma_\tau = \mathrm{var}\left[R^e_\tau\right]$ are respectively the mean vector and the variance-covariance matrix of risky asset excess returns, and $\mu_{\mathcal{D},\tau} = \mathbb{E}\left[R^e_\tau \mid \mathcal{D}_\tau\right]$ and $\Sigma_{\mathcal{D},\tau} = \mathrm{var}\left[R^e_\tau \mid \mathcal{D}_\tau\right]$ are respectively the mean vector and the variance-covariance matrix of risky asset excess returns conditional on the disappointing event.

The last equation in (C4) shows that the optimal solution may be written as

$$w = \frac{1}{\tilde{\gamma}_\tau}\left(\left(1-\alpha_{\mathcal{D},\tau}\right)\Sigma_\tau + \alpha_{\mathcal{D},\tau}\Sigma_{\mathcal{D},\tau}\right)^{-1}\left(\left(1-\lambda_{\mathcal{D},\tau}\right)\mu_\tau + \lambda_{\mathcal{D},\tau}\mu_{\mathcal{D},\tau}\right) = \frac{1}{\tilde{\gamma}_\tau}\tilde{\Sigma}_\tau^{-1}\tilde{\mu}_\tau$$

$$\text{where } \begin{cases} \tilde{\Sigma}_\tau = \left(1-\alpha_{\mathcal{D},\tau}\right)\Sigma_\tau + \alpha_{\mathcal{D},\tau}\Sigma_{\mathcal{D},\tau} \ \text{ and } \ \tilde{\mu}_\tau = \left(1-\lambda_{\mathcal{D},\tau}\right)\mu_\tau + \lambda_{\mathcal{D},\tau}\mu_{\mathcal{D},\tau} \\ \text{with } \ \lambda_{\mathcal{D},\tau} = \dfrac{\ell\pi^*_{\mathcal{D},\tau}}{1+\ell\pi^*_{\mathcal{D},\tau}}, \ \ \tilde{\gamma}_\tau = \left(1-\lambda_{\mathcal{D},\tau}\right)\gamma_\tau + \lambda_{\mathcal{D},\tau}\gamma_{\mathcal{D},\tau} \ \text{ and } \ \alpha_{\mathcal{D},\tau} = \dfrac{\lambda_{\mathcal{D},\tau}\gamma_{\mathcal{D},\tau}}{\tilde{\gamma}_\tau}. \end{cases} \tag{C5}$$

To obtain the optimal portfolio decomposition into key components, observe that:

$$\begin{aligned} \tilde{\Sigma}_\tau^{-1} &= \left(\left(1-\alpha_{\mathcal{D},\tau}\right)\Sigma_\tau + \alpha_{\mathcal{D},\tau}\Sigma_{\mathcal{D},\tau}\right)^{-1} = \Sigma_\tau^{-1} + \left[\left(\left(1-\alpha_{\mathcal{D},\tau}\right)\Sigma_\tau + \alpha_{\mathcal{D},\tau}\Sigma_{\mathcal{D},\tau}\right)^{-1} - \Sigma_\tau^{-1}\right] \\ &= \Sigma_\tau^{-1} + \left[\left(\left(1-\alpha_{\mathcal{D},\tau}\right)\Sigma_\tau + \alpha_{\mathcal{D},\tau}\Sigma_{\mathcal{D},\tau}\right)^{-1}\left\{\Sigma_\tau - \left(\left(1-\alpha_{\mathcal{D},\tau}\right)\Sigma_\tau + \alpha_{\mathcal{D},\tau}\Sigma_{\mathcal{D},\tau}\right)\right\}\Sigma_\tau^{-1}\right] \\ &= \Sigma_\tau^{-1} - \alpha_{\mathcal{D},\tau}\left(\left(1-\alpha_{\mathcal{D},\tau}\right)\Sigma_\tau + \alpha_{\mathcal{D},\tau}\Sigma_{\mathcal{D},\tau}\right)^{-1}\left(\Sigma_{\mathcal{D},\tau} - \Sigma_\tau\right)\Sigma_\tau^{-1}. \end{aligned}$$

Substituting out into equation (C5), we obtain the following optimal portfolio decomposition:

$$\begin{aligned} w = &\left(\frac{1}{\tilde{\gamma}_\tau}\Sigma_\tau^{-1}\mu_\tau\right) + \lambda_{\mathcal{D},\tau}\left(\frac{1}{\tilde{\gamma}_\tau}\Sigma_\tau^{-1}\left(\mu_{\mathcal{D},\tau} - \mu_\tau\right)\right) \\ &- \alpha_{\mathcal{D},\tau}\left[\Sigma_\tau + \alpha_{\mathcal{D},\tau}\left(\Sigma_{\mathcal{D},\tau} - \Sigma_\tau\right)\right]^{-1}\left(\Sigma_{\mathcal{D},\tau} - \Sigma_\tau\right)\left(\frac{1}{\tilde{\gamma}_\tau}\Sigma_\tau^{-1}\mu_\tau\right) \\ &- \lambda_{\mathcal{D},\tau}\alpha_{\mathcal{D},\tau}\left[\Sigma_\tau + \alpha_{\mathcal{D},\tau}\left(\Sigma_{\mathcal{D},\tau} - \Sigma_\tau\right)\right]^{-1}\left(\Sigma_{\mathcal{D},\tau} - \Sigma_\tau\right)\left(\frac{1}{\tilde{\gamma}_\tau}\Sigma_\tau^{-1}\left(\mu_{\mathcal{D},\tau} - \mu_\tau\right)\right). \end{aligned} \tag{C6}$$

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Table 1: **Summary statistics of asset excess returns**

$\tau = 1$	**RF**	**SK**	**CY**	**BD**	**RE**
Mean	0.007	0.029	0.013	0.013	0.023
Std		1.123	1.348	0.446	1.490
Ratio		0.026	0.009	0.030	0.015
Skew		−0.209	−0.498	−0.095	0.243
XKurt		11.018	7.585	3.058	27.565
Min		−11.985	−16.850	−2.804	−18.679
Q05		−1.718	−2.123	−0.710	−1.849
Q50		0.021	0.000	0.009	0.002
Q95		1.623	2.106	0.699	1.773
Max		11.579	7.909	4.133	18.715
Correlations					
CY		0.188			
BD		−0.193	−0.164		
RE		0.662	0.125	−0.104	

$\tau = 5$	**RF**	**SK**	**CY**	**BD**	**RE**
Mean	0.037	0.139	0.063	0.066	0.102
Std		2.311	2.995	0.987	2.983
Ratio		0.060	0.021	0.067	0.034
Skew		−0.526	−0.348	−0.075	0.030
XKurt		6.035	3.714	1.308	15.853
Min		−18.349	−20.115	−4.272	−31.362
Q05		−3.625	−4.770	−1.550	−4.310
Q50		0.274	0.117	0.099	0.207
Q95		3.502	4.548	1.609	4.001
Max		17.973	20.031	5.810	38.822
Correlations					
CY		0.230			
BD		−0.134	−0.177		
RE		0.663	0.149	−0.024	

$\tau = 10$	**RF**	**SK**	**CY**	**BD**	**RE**
Mean	0.075	0.270	0.123	0.132	0.195
Std		3.112	4.199	1.366	4.012
Ratio		0.087	0.029	0.096	0.049
Skew		−0.794	−0.495	0.057	−0.762
XKurt		5.643	3.100	1.475	9.670
Min		−25.887	−32.885	−5.497	−33.705
Q05		−4.871	−6.664	−2.117	−6.071
Q50		0.519	0.306	0.175	0.413
Q95		4.726	6.472	2.217	5.390
Max		21.632	26.394	8.526	30.031
Correlations					
CY		0.228			
BD		−0.129	−0.212		
RE		0.656	0.145	0.026	

$\tau = 20$	**RF**	**SK**	**CY**	**BD**	**RE**
Mean	0.149	0.534	0.259	0.265	0.388
Std		4.332	6.105	1.987	5.650
Ratio		0.123	0.042	0.134	0.069
Skew		−0.812	−0.434	0.201	−0.660
XKurt		4.982	2.698	1.625	7.862
Min		−30.981	−38.461	−6.768	−42.711
Q05		−6.752	−9.745	−3.024	−8.167
Q50		0.955	0.518	0.321	0.850
Q95		6.648	9.682	3.233	7.698
Max		26.174	31.920	11.486	40.567
Correlations					
CY		0.206			
BD		−0.123	−0.265		
RE		0.668	0.132	0.051	

Note: The top panel of the table presents, for different investment horizons (τ in days), sample values of the mean (Mean), standard deviation (Std), Sharpe ratio (Ratio), skewness (Skew), excess kurtosis (XKurt), minimum (Min), fifth percentile (Q05), fiftieth percentile (Q50), ninety-fifth percentile (Q95), and maximun (Max) for the risk-free rate (**RF**) and individual risky asset excess returns (**SK** for stock, **CY** for commodity, **BD** for bond, and **RE** for real estate). The bottom panel presents excess returns correlations. The mean, standard deviation, minimum, percentiles, and maximum are in percentage units. The data are daily, and the sample period is from January 2, 1989 to October 31, 2022.

Table 2: **Optimal portfolio weights and certainty equivalent**

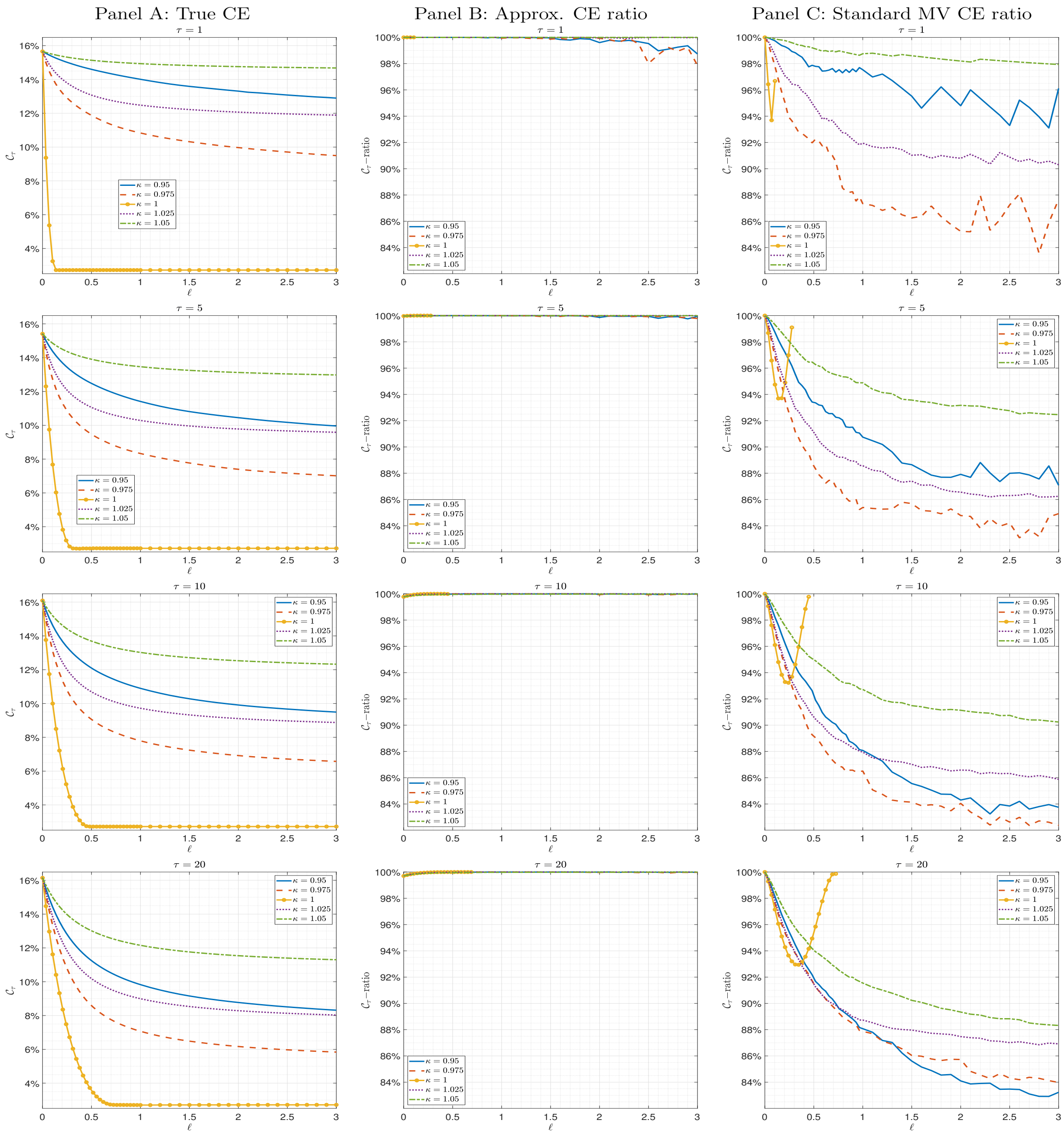
$\gamma\,[\ell]$		3[0]		3[1]					3[2]									3[3]				
κ				0.95	0.95	1	1.05	1.05	0.95	0.95	0.975	0.975	1	1.025	1.025	1.05	1.05	0.95	0.95	1	1.05	1.05
		(N)	(Y)	(N)	(Y)	(N)	(N)	(Y)	(N)	(Y)	(N)	(Y)	(N)	(N)	(Y)	(N)	(Y)	(N)	(Y)	(N)	(N)	(Y)
$\tau = 1$	**SK**	100.08	99.17	82.73	82.39	0.00	88.64	88.11	74.94	74.44	45.65	45.77	0.00	58.21	58.01	86.22	85.66	72.69	72.48	0.00	85.54	84.98
	CY	23.41	23.60	18.54	18.65	0.00	21.55	21.53	16.25	16.51	6.38	6.44	0.00	16.07	16.11	21.52	21.59	16.21	16.26	0.00	20.75	20.84
	BD	275.21	274.54	224.79	226.24	0.00	251.66	252.15	202.99	202.47	123.99	123.74	0.00	179.17	178.74	247.60	247.00	182.40	183.11	0.00	244.87	244.40
	RE	−10.13	−9.58	−14.92	−14.84	0.00	−11.35	−10.88	−17.47	−17.13	−10.13	−10.25	0.00	−9.12	−9.04	−11.94	−11.39	−16.48	−17.75	0.00	−11.85	−11.36
	$\mathcal{C}_\tau$	15.66	15.66	14.01	14.01	2.72	14.93	14.93	13.31	13.31	9.97	9.97	2.72	12.07	12.07	14.76	14.76	12.90	12.89	2.72	14.68	14.68
	$\mathcal{C}'_\tau$	15.66	15.66	14.01	12.23		14.93	14.71	13.26	8.89	9.96	−19.07		12.06	6.42	14.76	14.40	12.74	5.66		14.68	14.24
$\tau = 5$	**SK**	110.54	108.37	66.24	66.24	0.00	80.91	80.74	57.42	56.62	32.25	32.40	0.00	46.37	46.26	76.27	75.99	51.42	51.42	0.00	74.09	73.99
	CY	20.22	20.98	9.46	9.46	0.00	18.13	18.33	5.02	4.91	3.85	4.08	0.00	12.41	12.40	17.50	17.68	2.40	2.40	0.00	17.72	17.77
	BD	264.74	259.79	157.00	157.00	0.00	203.60	202.65	135.30	134.95	78.26	77.72	0.00	120.32	120.14	193.76	192.66	128.72	128.72	0.00	189.83	188.30
	RE	−21.79	−20.13	−16.74	−16.74	0.00	−15.37	−14.72	−17.47	−16.84	−9.10	−9.18	0.00	−6.90	−6.74	−14.64	−13.95	−16.81	−16.81	0.00	−14.03	−13.65
	$\mathcal{C}_\tau$	15.41	15.41	11.41	11.41	2.72	13.46	13.46	10.45	10.45	7.40	7.40	2.72	9.78	9.78	13.12	13.12	9.96	9.96	2.72	12.98	12.98
	$\mathcal{C}'_\tau$	15.41	15.41	11.41	3.60		13.46	12.08	10.44	−6.33	7.40	−29.67		9.78	−3.84	13.12	10.94	9.95	−14.57		12.98	10.38
$\tau = 10$	**SK**	124.53	118.62	64.37	64.04	0.00	81.32	80.20	55.73	55.61	31.41	31.63	0.00	45.59	45.60	73.67	73.41	52.69	52.49	0.00	70.92	70.49
	CY	22.21	23.81	10.79	11.23	0.00	19.06	19.60	7.41	7.31	4.65	4.52	0.00	11.91	12.01	18.88	19.25	4.40	4.57	0.00	18.70	19.03
	BD	281.72	269.65	145.00	144.88	0.00	195.39	192.29	120.74	120.04	68.64	68.24	0.00	112.57	112.01	182.56	180.46	111.78	111.81	0.00	177.51	175.17
	RE	−34.41	−28.70	−21.14	−20.55	0.00	−18.92	−17.25	−21.76	−21.58	−11.32	−11.45	0.00	−8.36	−8.14	−16.37	−15.29	−22.98	−22.77	0.00	−15.48	−14.53
	$\mathcal{C}_\tau$	16.09	16.05	10.90	10.90	2.72	13.02	13.02	9.91	9.91	6.92	6.92	2.72	9.11	9.11	12.53	12.53	9.49	9.49	2.72	12.33	12.32
	$\mathcal{C}'_\tau$	16.05	16.05	10.90	0.29		13.02	10.74	9.91	−11.54	6.92	−27.67		9.11	−6.59	12.53	8.87	9.49	−20.55		12.32	7.91
$\tau = 20$	**SK**	129.01	121.22	59.03	58.82	0.22	78.48	77.71	47.93	47.86	27.29	27.34	0.01	43.26	43.13	69.56	68.88	44.45	44.42	0.00	65.60	65.51
	CY	27.81	29.16	12.60	12.80	0.05	19.85	20.19	7.92	7.90	5.07	5.07	0.00	11.77	11.95	18.81	19.04	5.96	6.01	0.00	18.57	18.70
	BD	274.61	260.47	123.72	123.49	0.52	174.79	172.16	95.95	95.85	55.30	55.31	0.03	98.29	97.99	158.28	156.66	84.18	84.32	0.00	151.28	150.53
	RE	−40.39	−34.02	−20.96	−20.75	−0.08	−20.61	−19.43	−17.99	−17.89	−10.42	−10.47	−0.01	−9.09	−8.87	−17.45	−16.68	−16.47	−16.40	0.00	−16.45	−16.14
	$\mathcal{C}_\tau$	16.15	16.10	9.83	9.83	2.71	12.15	12.15	8.77	8.77	6.17	6.17	2.71	8.29	8.29	11.54	11.54	8.32	8.32	2.72	11.30	11.30
	$\mathcal{C}'_\tau$	16.10	16.10	9.83	−0.92		12.15	9.08	8.77	−12.67	6.17	−22.52		8.29	−6.71	11.54	6.39	8.31	−21.18		11.30	4.96

Note: For different investment horizons ($\tau \in \{1, 5, 10, 20\}$ in days) and for different combinations of preference parameters ($\gamma = 3$, $\ell \in \{0, 1, 2, 3\}$, $\kappa \in \{0.95, 0.975, 1, 1.025, 1.05\}$), the table displays estimates of the optimal portfolio weights (w) in individual risky assets (**SK** for stock, **CY** for commodity, **BD** for bond, and **RE** for real estate), and the associated certainty equivalent, $\mathcal{C}_\tau$, without the approximation (N) and with the approximation (Y). The cell $\mathcal{C}'_\tau$(N) is the certainty equivalent of the right-hand side of equation (19), while $\mathcal{C}'_\tau$(Y) corresponds to the certainty equivalent of the standard MV portfolio with equal endogenous risk aversion. The asset menu comprises the risk-free rate and four indices including stock, commodity, bond, and real estate. The data are daily, and the sample period is from January 2, 1989 to October 31, 2022.

Table 3: **Optimal portfolio decomposition**

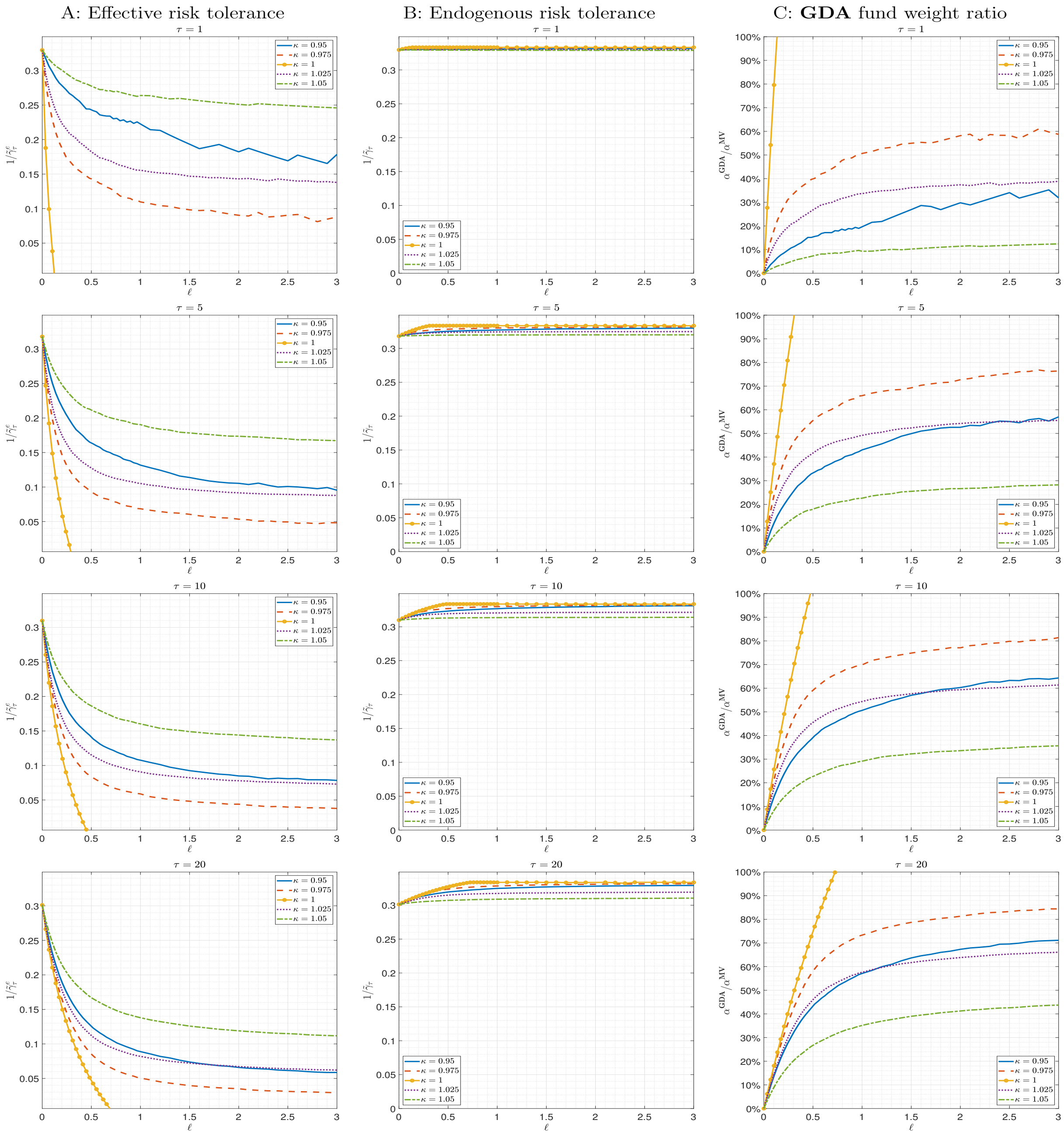
$\gamma\,[\ell]$		3[0]	3[1]				3[2]								3[3]			
κ			0.95		1.05		0.95		0.975		1.025		1.05		0.95		1.05	
$\bar{w}$		MV	(RDV)	**GDA**	(RDV)	**GDA**	(RDV)	**GDA**	(RDV)	**GDA**	(RDV)	**GDA**	(RDV)	**GDA**	(RDV)	**GDA**	(RDV)	**GDA**
$\tau = 1$	**SK**	25.58	528.96	−22.29	−4.97	−30.31	666.02	−22.74	478.47	−24.14	−8.39	−28.43	−4.76	−30.24	317.73	−17.16	−4.42	−29.10
	CY	6.09	424.09	−6.52	−0.70	−5.01	534.33	−6.51	390.01	−7.68	−1.44	−5.31	−0.65	−4.40	783.56	−5.70	−0.62	−5.63
	BD	70.81	442.82	−64.58	−2.18	−61.00	673.95	−63.29	257.31	−68.08	−4.82	−66.13	−2.11	−61.13	157.02	−70.69	−2.11	−61.53
	RE	−2.47	750.12	−6.61	−4.38	−3.68	889.87	−7.46	1,361.97	−0.09	−7.18	−0.13	−4.29	−4.23	593.29	−6.44	−4.04	−3.74
	$\mathbf{1}^\top\bar{w}$	100.00		−100.00		−100.00		−100.00		−100.00		−100.00		−100.00		−100.00		−100.00
	α	387.72	389.24	76.43	387.11	35.93	389.85	116.09	391.05	227.46	388.02	145.10	386.94	44.15	390.18	124.40	386.87	48.00
$\tau = 5$	**SK**	29.37	303.61	−27.66	−8.30	−33.59	366.13	−27.50	299.94	−28.68	−10.08	−31.51	−7.77	−33.38	368.52	−28.92	−7.55	−33.42
	CY	5.68	351.21	−7.25	−1.74	−3.26	473.00	−7.86	343.79	−6.44	−2.23	−4.29	−1.58	−3.53	375.94	−8.80	−1.44	−3.18
	BD	70.40	48.00	−67.61	−6.41	−69.76	45.88	−66.85	28.39	−69.05	−9.24	−71.01	−5.88	−69.46	36.57	−64.72	−5.76	−69.70
	RE	−5.45	463.82	2.52	−6.90	6.61	528.50	2.21	465.71	4.16	−8.42	6.82	−6.92	6.37	694.90	2.45	−6.92	6.29
	$\mathbf{1}^\top\bar{w}$	100.00		−100.00		−100.00		−100.00		−100.00		−100.00		−100.00		−100.00		−100.00
	α	369.02	379.57	163.31	371.04	83.88	381.77	200.75	384.16	279.28	376.69	204.17	371.29	98.94	382.71	218.11	371.32	104.90
$\tau = 10$	**SK**	30.94	201.11	−29.83	−9.84	−35.20	310.91	−28.79	224.42	−30.03	−10.45	−32.79	−9.28	−35.80	407.00	−28.18	−9.12	−36.00
	CY	6.21	192.01	−6.77	−0.74	−3.97	282.83	−7.23	218.83	−6.50	−1.29	−5.39	−0.80	−3.72	365.87	−7.92	−0.86	−3.61
	BZ	70.33	−1.06	−68.15	−9.39	−71.22	−3.85	−67.87	−12.04	−69.69	−10.77	−71.02	−8.66	−71.03	−2.73	−66.90	−8.39	−70.82
	RE	−7.49	334.91	4.75	−4.77	10.39	482.29	3.89	370.36	6.22	−5.00	9.20	−4.73	10.55	627.01	2.99	−4.52	10.42
	$\mathbf{1}^\top\bar{w}$	100.00		−100.00		−100.00		−100.00		−100.00		−100.00		−100.00		−100.00		−100.00
	α	383.38	404.26	204.61	387.94	113.05	407.99	245.92	410.18	316.16	397.46	235.86	388.28	130.37	409.60	263.49	388.48	138.47
$\tau = 20$	**SK**	32.17	126.14	−30.99	−12.63	−34.32	170.25	−30.41	132.30	−31.34	−12.71	−33.45	−11.90	−34.95	192.46	−29.85	−11.56	−35.00
	CY	7.74	113.46	−8.04	−1.98	−7.16	193.36	−8.59	127.26	−8.09	−1.57	−7.43	−1.83	−6.91	212.50	−8.85	−1.89	−6.67
	BD	69.12	11.14	−67.83	−12.84	−69.90	6.54	−68.00	−1.51	−68.65	−13.30	−69.78	−11.88	−69.63	−4.77	−68.31	−11.45	−69.44
	RE	−9.03	284.60	6.86	−7.43	11.39	383.67	6.99	319.75	8.08	−7.03	10.66	−6.41	11.48	434.31	7.01	−5.99	11.10
	$\mathbf{1}^\top\bar{w}$	100.00		−100.00		−100.00		−100.00		−100.00		−100.00		−100.00		−100.00		−100.00
	α	376.83	406.41	232.10	386.39	135.78	410.57	276.40	414.07	336.60	398.66	254.48	387.66	159.84	411.79	292.88	388.40	169.82

Note: For different investment horizons ($\tau \in \{1, 5, 10, 20\}$ in days) and for different combinations of preference parameters ($\gamma = 3$, $\ell \in \{0, 1, 2, 3\}$, $\kappa \in \{0.95, 0.975, 1.025, 1.05\}$), the table displays estimates of the weights in individual risky assets (**SK** for stock, **CY** for commodity, **BD** for bond, and **RE** for real estate) and their sum, for each portfolio component ($\bar{w}^{\mathbf{MV}}$ and $\bar{w}^{\mathbf{GDA}}$), as well as the weight of each component in the optimal portfolio ($\alpha^{\mathbf{MV}}$ and $\alpha^{\mathbf{GDA}}$). Estimates of the individual asset relative downside variance in percentage are provided in columns labeled (RDV), followed by $\alpha^{\mathbf{MV}}$. The asset menu comprises the risk-free rate and four indices (stock, commodity, bond, and real estate). The data are daily, from January 2, 1989 to October 31, 2022.



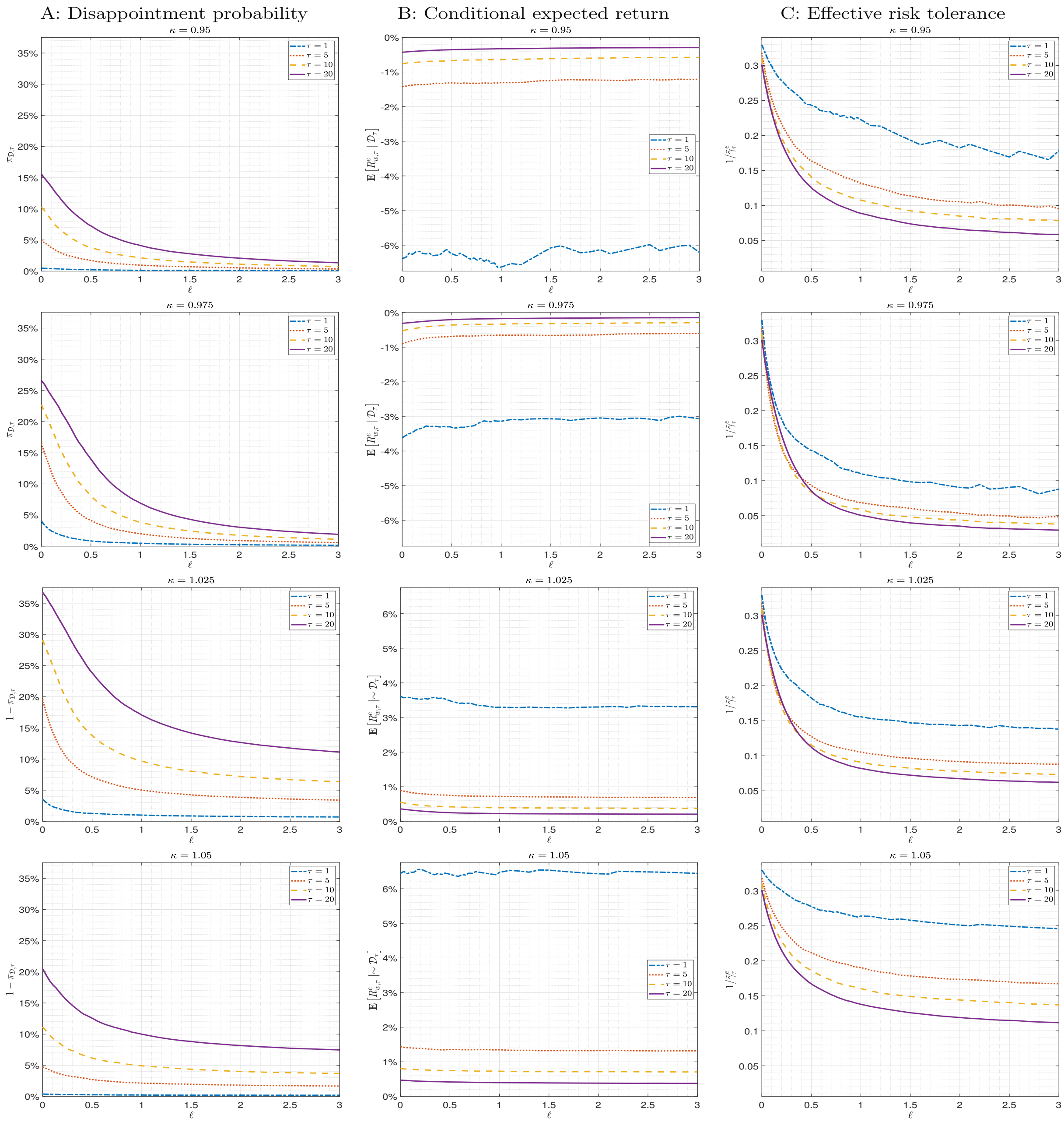
Note: The figure shows estimates of the certainty equivalent (CE) of the optimal portfolio ($\mathcal{C}_\tau$) for different investment horizons ($\tau \in \{1, 5, 10, 20\}$ days) and preference parameters ($\gamma = 3$, $\ell \in [0, 3]$, $\kappa \in \{0.95, 0.975, 1, 1.025, 1.05\}$). Panel A presents results without approximation. Panel B shows the ratio of the CE with vs. without approximation. Panel C compares the standard MV portfolio's CE to the approximated version. The asset menu includes a risk-free rate and four indices (stock, commodity, bond, real estate), with daily data from January 2, 1989, to October 31, 2022.

Figure 1: **Certainty Equivalent: Optimality and accuracy of approximations**



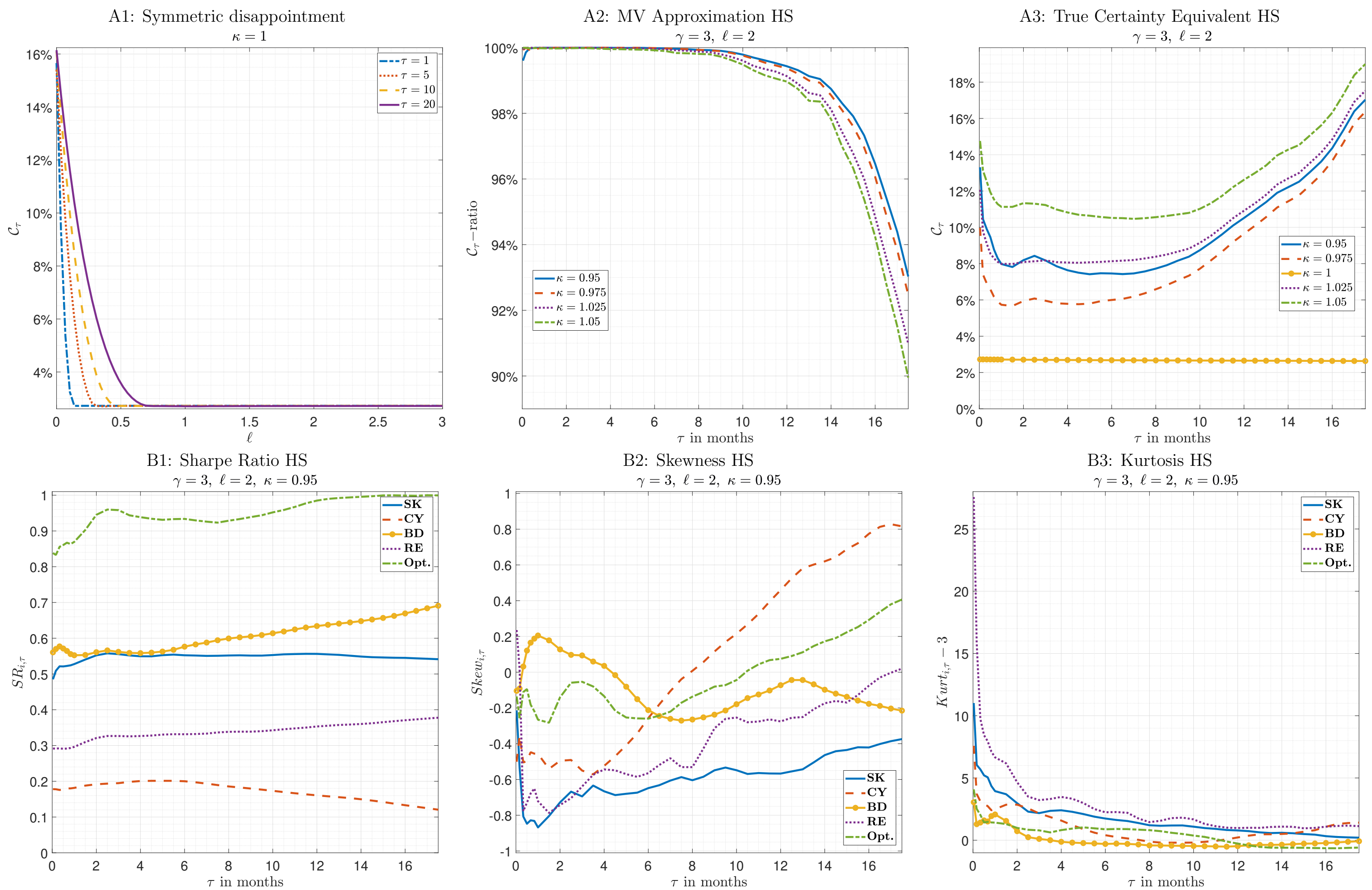
Note: For different investment horizons ($\tau \in \{1, 5, 10, 20\}$ days) and preference parameters ($\gamma = 3$, $\ell \in [0, 3]$, $\kappa \in \{0.95, 0.975, 1, 1.025, 1.05\}$), Panel A shows the effective risk tolerance $1/\tilde{\gamma}^e_\tau$, Panel B displays the endogenous risk tolerance $1/\tilde{\gamma}_\tau$, while Panel C plots the ratio $\alpha^{\mathbf{GDA}}/\alpha^{\mathbf{MV}}$ of the normalized **GDA** fund weight to the normalized **MV** fund weight. The asset menu includes a risk-free rate and four indices (stock, commodity, bond, real estate), with daily data from January 2, 1989, to October 31, 2022.

Figure 2: **Effective risk tolerance in approximate MV portfolio optimization**



Note: The figure shows, for different investment horizons ($\tau \in \{1, 5, 10, 20\}$ days) and preference parameters ($\gamma = 3$, $\ell \in [0, 3]$, $\kappa \in \{0.95, 0.975, 1.025, 1.05\}$), estimates of $\pi_{\mathcal{D},\tau}$ (the real-world disappointment probability for $\kappa < 1$) or $1 - \pi_{\mathcal{D},\tau}$ ($\kappa > 1$) in Panel A, and estimates of $\mathbb{E}\left[R^e_{w,\tau} \mid \mathcal{D}_\tau\right]$ (downside expected excess return for $\kappa < 1$) or $\mathbb{E}\left[R^e_{w,\tau} \mid\sim \mathcal{D}_\tau\right]$ ($\kappa > 1$) in Panel B, where $\sim \mathcal{D}_\tau$ is the complement of $\mathcal{D}_\tau$. The asset menu includes a risk-free rate and four indices (stock, commodity, bond, real estate), with daily data from January 2, 1989, to October 31, 2022.

Figure 3: **Disappointment probabilities and conditional expected returns**



Note: Panel A1 plots the certainty equivalent (CE) of the optimal portfolio, computed without approximation, across investment horizons ($\tau \in \{1, 5, 10, 20\}$ days) for GDA preferences with $\gamma = 3$, $\kappa = 1$, and varying $\ell \in [0, 3]$. The remaining panels fix $\gamma = 3$ and $\ell = 2$ and examine the horizon structure (HS) of key quantities: Panel A2 shows the ratio of CE with vs. without approximation; Panel A3 shows the CE without approximation; Panel B1 displays Sharpe ratios; Panel B2, skewness; and Panel B3, excess kurtosis—for both the individual assets and the optimal portfolio. The asset universe consists of a risk-free rate and four asset classes (stock, commodity, bond, and real estate), using daily data from January 2, 1989, to October 31, 2022.

Figure 4: **Additional horizon effects**

INTERNET APPENDIX of **"Endogenous Downside Risk and Asset Allocation"**

This supplemental appendix for "Endogenous Downside Risk and Asset Allocation" discusses asset and portfolio performance measurement motivated by our nonstandard mean-variance framework to suggest how practitioners can use or implement our setting. It also provides additional tables and figures that complement the analysis presented in the main text.

A Portfolio performance measures

In the article's main body, we demonstrate that generalized disappointment aversion (GDA) significantly alters optimal asset allocation, reshapes effective risk aversion, and induces horizon-dependent investment behavior. These theoretical insights have a direct impact on portfolio performance, particularly when downside risk is a salient factor. While our framework is grounded in GDA preferences, it naturally leads to new performance metrics that extend and reinterpret well-established measures used in practice.

The literature offers numerous criteria to assess a portfolio's return-risk trade-off, most of which are designed to reflect investor preferences. Classical measures such as the Sharpe ratio, Jensen's alpha, and the Treynor index originate from mean-variance theory and implicitly assume expected utility preferences. In contrast, a growing set of alternative metrics explicitly incorporate downside risk aversion. These include the gain-loss ratio of Bernardo and Ledoit (2000)—later popularized as the omega ratio by Keating and Shadwick (2002)—as well as the Sortino ratio (Sortino and van der Meer; 1991) and the upside potential ratio introduced by Sortino et al. (1999).

Another example is the Conditional Sharpe Ratio (CSR), which replaces standard deviation with expected shortfall (ES) or conditional value-at-risk (CVaR) to account for tail risk. Widely used in hedge fund and private equity contexts, CSR is particularly relevant when protecting against large downside losses and return distributions exhibit significant skewness or kurtosis.

Our GDA-based framework complements these approaches by providing a formal foundation for performance evaluation under asymmetric, kinked utility. Specifically, equation (14) shows that the downside expected return of an asset is computed conditional on the investor's overall portfolio falling below her disappointment benchmark—a concept closely related to the marginal expected shortfall employed in systemic risk analysis (e.g., Acharya et al.; 2013, 2016). As such, disappointment-averse investors endogenously incorporate downside risks into their portfolio decisions, leading to novel return-risk trade-offs.

In what follows, we introduce an investment performance measure implied by GDA preferences and relate it to existing downside-sensitive metrics. This measure—the endogenous Sharpe ratio—captures the adjustment of both expected returns and variances to account for disappointment risk, providing a unified framework for performance evaluation under behavioral preferences.

We consider the following decomposition of the asset excess returns R^e_τ into a downside (a loss) component L_τ and an upside (a gain) component G_τ:

$$R^e_\tau = G_\tau - L_\tau \quad \text{where} \quad L_\tau = -R^e_\tau \mathbb{I}\left(\mathcal{D}_\tau\right) \quad \text{and} \quad G_\tau = R^e_\tau \left(1 - \mathbb{I}\left(\mathcal{D}_\tau\right)\right), \tag{A.1}$$

and where L_τ and G_τ represent the downside (loss) and the upside (gain) components of the excess returns, respectively. Since positive downside (loss) and upside (gain) components cannot coincide, we observe that (the element-by-element product) $L_\tau \cdot G_\tau = 0$. This upside-downside (gain-loss) decomposition of asset returns is exploited in an approach to asset pricing in incomplete markets by Bernardo and Ledoit (2000), where the ratio of the risk-adjusted expected upside to the risk-adjusted expected downside excess return, called the gain-loss ratio, summarizes the attractiveness of any zero-cost portfolio. We define:

$$\mu^-_{\mathcal{D},\tau} \equiv \mathbb{E}\left[L_\tau\right] \quad \text{and} \quad \mu^+_{\mathcal{D},\tau} \equiv \mathbb{E}\left[G_\tau\right]. \tag{A.2}$$

Using these moments, the standard Sharpe ratio, $SR_{i,\tau}$, and analogues of the omega ratio, $OR_{i,\tau}$, Sortino ratio, $TR_{i,\tau}$, upside potential ratio, $UR_{i,\tau}$, and conditional Sharpe ratio, $CSR_{i,\tau}$, given the disappointing event $\mathcal{D}_\tau$ would be

$$SR_{i,\tau} = \frac{\mu_{i,\tau}}{\sigma_{i,\tau}}, \;\; OR_{i,\tau} = \frac{\mu^{+}_{i\mathcal{D},\tau}}{\mu^{-}_{i\mathcal{D},\tau}}, \;\; TR_{i,\tau} = \frac{\mu_{i,\tau}}{\sigma_{i\mathcal{D},\tau}}, \;\; UR_{i,\tau} = \frac{\mu^{+}_{i\mathcal{D},\tau}}{\sigma_{i\mathcal{D},\tau}} \text{ and } CSR_{i,\tau} = \frac{\mu_{i,\tau}}{-\mu_{i\mathcal{D},\tau}}, \tag{A.3}$$

where $-\mu_{i\mathcal{D},\tau}$ measures the asset's marginal expected shortfall, coinciding with the expected shortfall when the asset is the portfolio itself.

Our GDA-based framework complements existing performance evaluation methods by offering a theoretical foundation tailored to asymmetric, kinked preferences. We have shown that disappointment-averse investors endogenously distort return and risk measures, assigning an optimal weight to downside moments. These preference-adjusted moments give rise to a new class of return-risk metrics that capture both conventional and disappointment-induced trade-offs. In particular, the endogenous mean-variance interpretation of optimal portfolio choice (see equation (16)) leads naturally to a new performance measure, which we refer to as the *endogenous Sharpe ratio*, $\widetilde{SR}_{i,\tau}$, defined as:

$$\begin{aligned}\widetilde{SR}_{i,\tau} = \frac{\tilde{\mu}_{i,\tau}}{\tilde{\sigma}_{i,\tau}} &= \frac{(1-\lambda_{\mathcal{D},\tau})\mu_{i,\tau} + \lambda_{\mathcal{D},\tau}\mu_{i\mathcal{D},\tau}}{\sqrt{(1-\alpha_{\mathcal{D},\tau})\sigma^2_{i,\tau} + \alpha_{\mathcal{D},\tau}\sigma^2_{i\mathcal{D},\tau}}}, \quad i = w, 1, 2, \ldots, n \\ &= SR_{i,\tau} \cdot \frac{(1-\lambda_{\mathcal{D},\tau}) + \lambda_{\mathcal{D},\tau}\frac{\mu_{i\mathcal{D},\tau}}{\mu_{i,\tau}}}{\sqrt{(1-\alpha_{\mathcal{D},\tau}) + \alpha_{\mathcal{D},\tau}\frac{\sigma^2_{i\mathcal{D},\tau}}{\sigma^2_{i,\tau}}}}.\end{aligned} \tag{A.4}$$

Computing $\widetilde{SR}_{i,\tau}$ requires specification of a downside event $\mathcal{D}$ and associated weights: $\lambda_{\mathcal{D},\tau}$ for the downside expected return, and $\alpha_{\mathcal{D},\tau}$ for the downside variance. Investor preferences endogenously determine these weights. For instance, Figure A1-A plots $\lambda_{\mathcal{D},\tau}$ across different horizons for selected preference parameters. Similarly, Figure A2-A does the same for $\alpha_{\mathcal{D},\tau}$. These quantities vary systematically with the horizon: for $\kappa < 1$, both weights increase modestly with the horizon (typically remaining below 7% at $\ell = 2$); for $\kappa > 1$, they are relatively stable (around 65% at $\ell = 2$).

In practical settings, the precise values of $\lambda_{\mathcal{D},\tau}$ and $\alpha_{\mathcal{D},\tau}$ may not be known ex-ante. Nonetheless, the framework remains operational using ad hoc but reasonable reference values. Practitioners can select downside events based on scenario analysis—e.g., a common macro or market shock, such as the GDP, the industrial production index, or the stock index falling below its first quartile, or portfolio-specific events, such as a drop below the portfolio's historical median return. Likewise, reasonable fixed weights—e.g., one-fifth on the downside expected return and one-fourth on the downside variance—can be used for benchmarking purposes.

Notably, the ratio $\alpha_{\mathcal{D},\tau}/\lambda_{\mathcal{D},\tau}$ tends to lie between 1 and 1.2, which means practitioners may use a single common penalty coefficient on both downside moments without materially distorting the endogenous Sharpe ratio. This robustness facilitates ease of use in applied settings. More broadly, because both the disappointing event and the downside penalty can be flexibly chosen, our framework lends itself naturally to scenario analysis. This flexibility makes it particularly useful for stress testing, custom benchmarking, and performance attribution under behavioral preferences prioritizing downside protection.

Equation (A.4) highlights two key inputs: the ratio of downside to standard expected excess returns and the ratio of downside to standard variances. These ratios adjust the standard Sharpe ratio to reflect the investor's aversion to downside risk, yielding the endogenous Sharpe ratio. When the downside expected excess return is relatively low (i.e., the ratio is below one) and the downside variance is relatively high (i.e., the ratio exceeds one), the endogenous Sharpe ratio falls below its standard counterpart. Conversely, a higher downside-to-standard expected return ratio and a lower downside-to-standard variance ratio improve the endogenous Sharpe ratio.

Importantly, the endogenous Sharpe ratio can also be reformulated in terms of widely used downside-sensitive performance measures:

$$\widetilde{SR}_{i,\tau} = SR_{i,\tau} \cdot \frac{(1-\lambda_{\mathcal{D},\tau}) - \lambda_{\mathcal{D},\tau}/CSR_{i,\tau}}{\sqrt{(1-\alpha_{\mathcal{D},\tau}) + \alpha_{\mathcal{D},\tau} \cdot \left(SR_{i,\tau}/TR_{i,\tau}\right)^2}}, \quad i = w, 1, 2, \ldots, n. \tag{A.5}$$

This expression shows that $\widetilde{SR}_{i,\tau}$ is a strictly increasing function of the standard Sharpe ratio, the Conditional Sharpe Ratio (CSR), and the Sortino Ratio (TR), holding all else constant. It offers a unified framework in which GDA preferences give rise to a performance measure that embeds—and extends—several familiar risk-adjusted metrics used to evaluate portfolios under downside risk aversion.

Figure A2-B shows that the annualized standard Sharpe ratio of the optimal portfolio is nearly invariant across preference configurations for a given investment horizon. It also rises steadily with the horizon within the short range we study (see Panel B1 of Figure 4 in the main article), consistent with improvements in risk-return trade-offs driven by statistical properties of asset returns. In contrast, the endogenous Sharpe ratio of assets and optimal portfolio—our preference-sensitive measure implemented by GDA—varies significantly between investors for a given horizon, as shown in Figure A1-B and Figure A1-C. This variation reflects how investors with different levels of disappointment aversion and asymmetry differentially weigh downside moments in asset and portfolio returns. Notably, the endogenous Sharpe ratio is consistently lower than its standard counterpart (corresponding to $\ell = 0$ on the graphs in Figure A1-B and Figure A1-C) for all GDA investors, highlighting the psychological cost of disappointment risk embedded in asset allocation. Naturally, the endogenous Sharpe ratios of the individual assets are all smaller than that of the optimal portfolios, consistent with the endogenous mean-variance framework, which is equivalent to maximizing the endogenous Sharpe ratio when preferences represented by the downside event and the downside weight are fixed.

This observation aligns with the performance measurement framework proposed by Cogneau and Hübner (2009a,b), who emphasize the limitations of applying standard risk-adjusted measures—such as the Sharpe ratio—uniformly across investors with heterogeneous preferences. In their critical taxonomy of over 100 performance metrics, the authors emphasize the importance of aligning performance evaluation with investors' specific risk sensitivities, particularly in the presence of concerns about downside risk. Our endogenous Sharpe ratio fits naturally within this class of preference-adjusted indicators. It preserves the intuitive appeal of the traditional Sharpe ratio while embedding it in a richer behavioral framework that adjusts inputs endogenously for disappointment aversion. As such, it offers theoretical consistency and practical relevance, providing a tailored, utility-consistent benchmark for assessing portfolio efficiency under nonstandard preferences, particularly in short-horizon investment contexts where psychological frictions are most salient.

B Additional tables and figures

Table A1: **Summary statistics of asset excess returns: Non-overlapping**

	$\tau = 1$				
	RF	**SK**	**CY**	**BD**	**RE**
Mean	0.007	0.029	0.013	0.013	0.023
Std		1.123	1.348	0.446	1.490
Ratio		0.026	0.009	0.030	0.015
Skew		−0.209	−0.498	−0.095	0.243
XKurt		11.018	7.585	3.058	27.565
Min		−11.985	−16.850	−2.804	−18.679
Q05		−1.718	−2.123	−0.710	−1.849
Q50		0.021	0.000	0.009	0.002
Q95		1.623	2.106	0.699	1.773
Max		11.579	7.909	4.133	18.715
		Correlations			
CY		0.188			
BD		−0.193	−0.164		
RE		0.662	0.125	−0.104	

	$\tau = 5$				
	RF	**SK**	**CY**	**BD**	**RE**
Mean	0.037	0.142	0.064	0.066	0.110
Std		2.449	3.023	1.010	3.216
Ratio		0.058	0.021	0.065	0.034
Skew		−0.171	−0.216	−0.140	0.872
XKurt		5.323	3.841	1.718	20.992
Min		−13.863	−17.467	−4.218	−22.216
Q05		−3.814	−4.915	−1.581	−4.623
Q50		0.247	0.154	0.124	0.191
Q95		3.688	4.577	1.615	3.924
Max		17.397	19.907	5.626	38.822
		Correlations			
CY		0.210			
BD		−0.145	−0.186		
RE		0.672	0.132	−0.049	

	$\tau = 10$				
	RF	**SK**	**CY**	**BD**	**RE**
Mean	0.075	0.277	0.124	0.130	0.195
Std		3.270	4.218	1.364	4.049
Ratio		0.085	0.029	0.095	0.048
Skew		−0.807	−0.637	−0.039	−0.692
XKurt		4.801	2.953	0.838	12.563
Min		−22.798	−25.127	−4.484	−33.705
Q05		−5.183	−6.476	−2.090	−5.951
Q50		0.505	0.231	0.187	0.390
Q95		5.011	6.593	2.220	5.379
Max		14.529	14.527	5.347	30.031
		Correlations			
CY		0.220			
BD		−0.154	−0.228		
RE		0.665	0.154	−0.005	

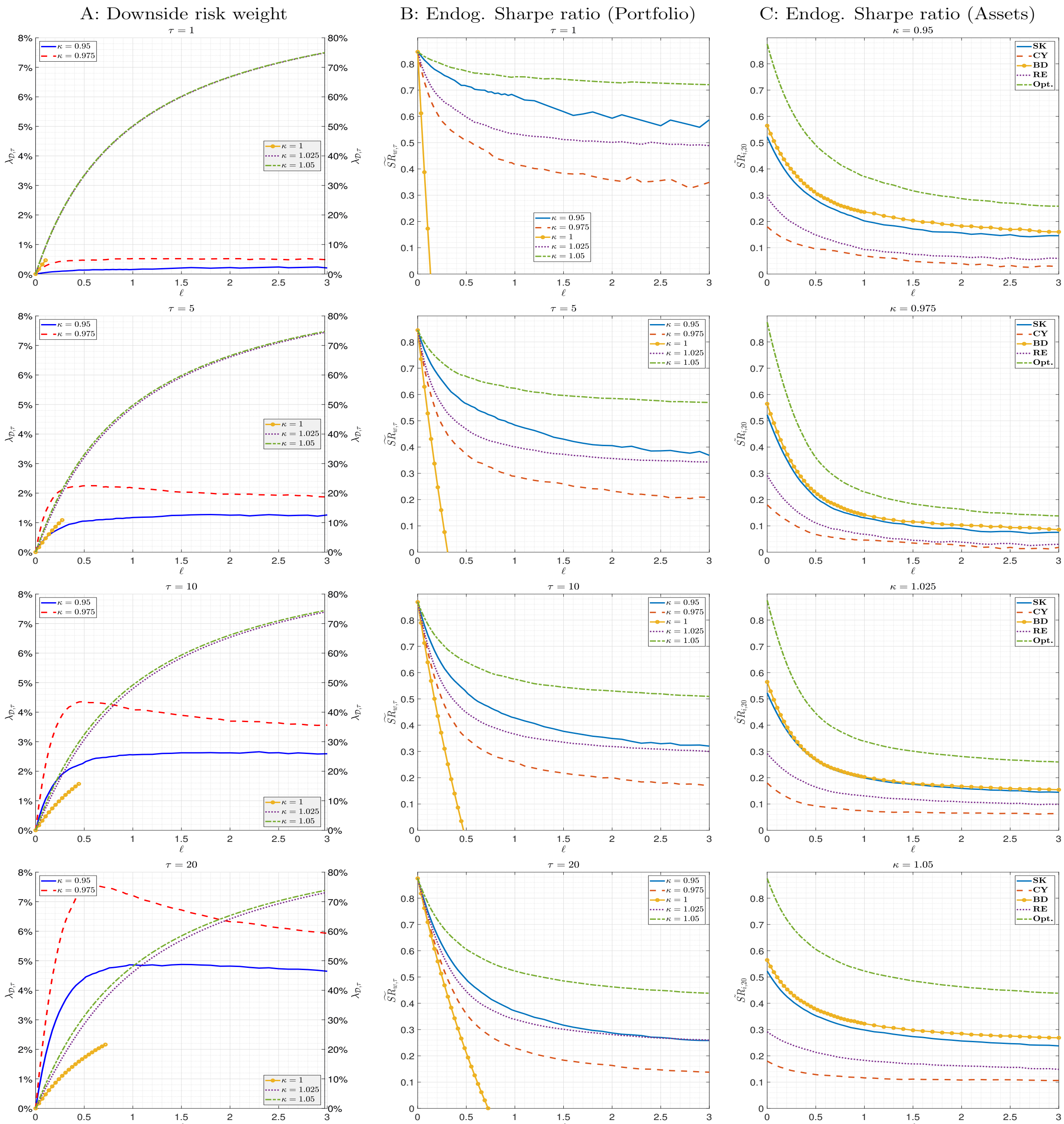
	$\tau = 20$				
	RF	**SK**	**CY**	**BD**	**RE**
Mean	0.149	0.550	0.256	0.261	0.380
Std		4.500	6.070	1.976	5.488
Ratio		0.122	0.042	0.132	0.069
Skew		−1.132	−0.590	0.165	−0.948
XKurt		5.587	1.769	1.979	7.759
Min		−29.461	−30.296	−5.958	−32.873
Q05		−7.489	−9.781	−3.048	−6.935
Q50		0.928	0.216	0.405	0.826
Q95		6.560	9.921	3.268	7.570
Max		15.731	15.795	9.973	29.568
		Correlations			
CY		0.234			
BD		−0.103	−0.247		
RE		0.641	0.166	0.008	

Note: The top panel of the table presents, for different investment horizons (τ in days), sample values of the mean (Mean), standard deviation (Std), Sharpe ratio (Ratio), skewness (Skew), excess kurtosis (XKurt), minimum (Min), fifth percentile (Q05), fiftieth percentile (Q50), ninety-fifth percentile (Q95), and maximun (Max) for the risk-free rate (**RF**) and individual risky asset excess returns (**SK** for stock, **CY** for commodity, **BD** for bond, and **RE** for real estate). The bottom panel presents excess returns correlations. The mean, standard deviation, minimum, percentiles, and maximum are in percentage units. The data are daily, and the sample period is from January 2, 1989 to October 31, 2022.

Table A2: **Optimal portfolio weights and certainty equivalent: Non-overlapping**

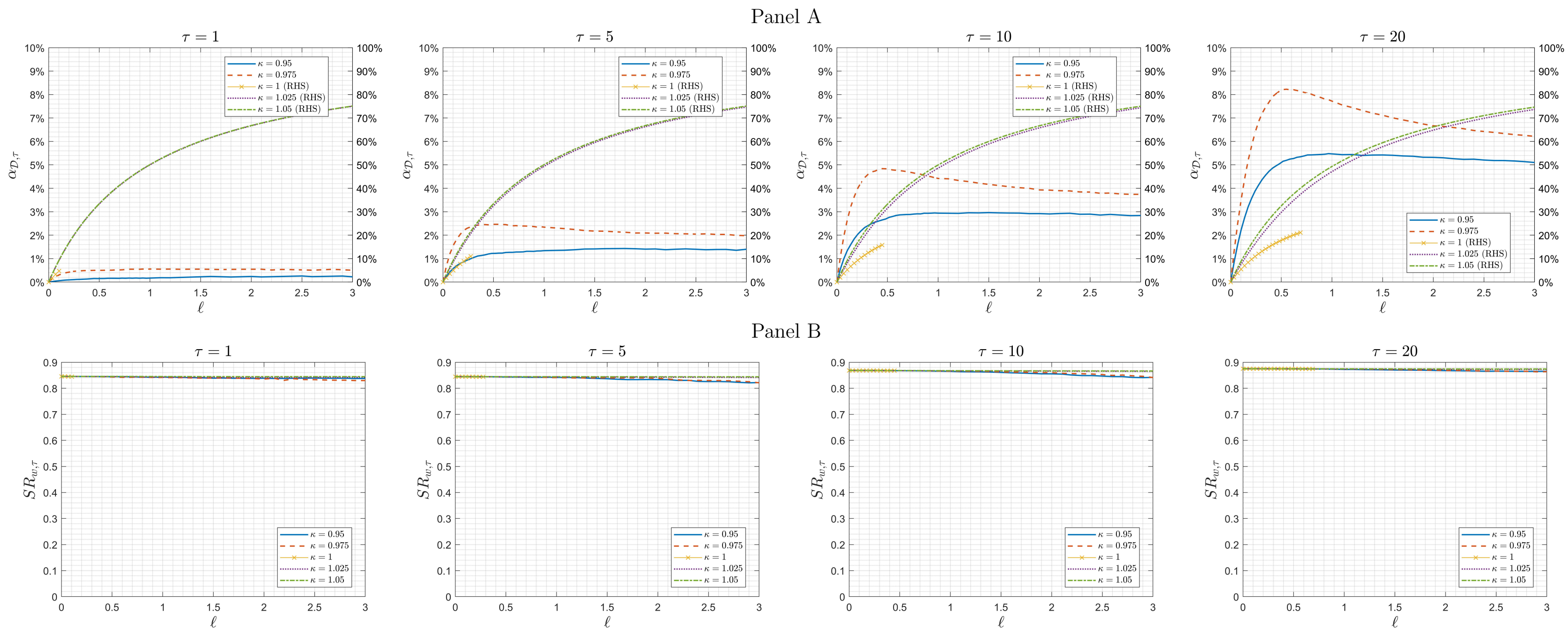
$\gamma\,[\ell]$		3[0]		3[1]					3[2]									3[3]				
κ				0.95	0.95	1	1.05	1.05	0.95	0.95	0.975	0.975	1	1.025	1.025	1.05	1.05	0.95	0.95	1	1.05	1.05
		(N)	(Y)	(N)	(Y)	(N)	(N)	(Y)	(N)	(Y)	(N)	(Y)	(N)	(N)	(Y)	(N)	(Y)	(N)	(Y)	(N)	(N)	(Y)
$\tau=1$	**SK**	100.08	99.17	82.73	82.39	0.00	88.64	88.11	75.20	74.44	45.65	45.77	0.00	57.69	58.12	86.22	85.66	72.57	72.57	0.00	85.54	84.98
	CY	23.41	23.60	18.54	18.65	0.00	21.55	21.53	16.16	16.68	6.38	6.44	0.00	16.07	15.88	21.52	21.59	16.63	16.63	0.00	20.75	20.84
	BD	275.21	274.54	224.79	226.24	0.00	251.66	252.15	208.10	202.48	123.99	123.74	0.00	178.30	178.66	247.60	247.00	182.54	182.54	0.00	244.87	244.40
	RE	−10.13	−9.58	−14.92	−14.84	0.00	−11.35	−10.88	−17.37	−17.20	−10.13	−10.25	0.00	−8.91	−9.06	−11.94	−11.39	−16.55	−16.55	0.00	−11.85	−11.36
	$\mathcal{C}_\tau$	15.66	15.66	14.01	14.01	2.72	14.93	14.93	13.31	13.31	9.97	9.97	2.72	12.07	12.07	14.76	14.76	12.90	12.90	2.72	14.68	14.68
	$\mathcal{C}'_\tau$	15.66	15.66	14.01	12.23		14.93	14.71	13.26	8.89	9.96	−19.07		12.06	6.42	14.76	14.40	12.74	5.66		14.68	14.24
$\tau=5$	**SK**	101.16	98.40	59.42	59.71	0.00	70.77	70.21	52.79	53.89	29.63	31.04	0.00	38.85	38.96	67.71	66.80	48.15	48.06	0.00	65.19	64.27
	CY	22.64	23.21	10.99	10.85	0.00	20.76	20.69	7.74	10.01	4.26	3.67	0.00	14.44	14.68	19.67	20.04	8.00	8.80	0.00	20.38	19.82
	BD	253.78	252.24	138.77	138.21	0.00	194.26	194.23	112.93	106.55	64.58	65.44	0.00	118.12	118.78	189.74	189.00	110.43	111.71	0.00	188.45	184.29
	RE	−16.39	−15.05	−12.36	−11.98	0.00	−10.86	−10.17	−14.41	−15.14	−6.14	−6.66	0.00	−4.17	−4.21	−9.91	−9.21	−17.22	−17.39	0.00	−8.36	−7.96
	$\mathcal{C}_\tau$	14.90	14.89	10.96	10.96	2.72	13.11	13.11	10.10	10.10	7.15	7.15	2.72	9.54	9.54	12.82	12.82	9.80	9.80	2.72	12.69	12.69
	$\mathcal{C}'_\tau$	14.89	14.89	10.96	3.49		13.11	11.65	10.06	−6.16	7.04	−28.36		9.54	−3.65	12.82	10.56	9.31	−14.21		12.68	10.02
$\tau=10$	**SK**	118.32	113.31	60.56	60.26	0.00	76.13	75.70	49.65	50.31	29.47	29.65	0.00	45.09	44.56	69.16	68.05	47.03	49.35	0.00	66.78	64.41
	CY	25.46	26.54	11.05	12.83	0.00	22.02	22.52	10.04	10.35	7.32	5.30	0.00	13.57	13.67	20.87	20.98	5.82	3.91	0.00	20.07	20.76
	BD	285.24	276.43	141.01	146.07	0.00	196.52	195.28	119.31	118.91	70.35	62.75	0.00	115.93	114.92	177.96	175.56	111.42	100.13	0.00	173.11	172.22
	RE	−32.30	−27.25	−21.69	−22.31	0.00	−15.46	−13.98	−20.03	−20.67	−12.19	−11.36	0.00	−7.10	−6.90	−13.47	−11.23	−18.91	−17.50	0.00	−12.23	−11.04
	$\mathcal{C}_\tau$	16.05	16.03	10.67	10.67	2.72	12.91	12.91	9.79	9.79	6.76	6.78	2.72	9.08	9.08	12.41	12.41	9.41	9.43	2.72	12.21	12.21
	$\mathcal{C}'_\tau$	16.03	16.03	10.62	−0.09		12.91	10.70	9.74	−11.97	6.65	−27.90		9.06	−6.42	12.41	8.82	9.09	−21.08		12.21	7.83
$\tau=20$	**SK**	107.25	99.96	49.01	48.59	0.00	64.07	62.95	42.48	39.51	23.78	24.27	0.00	36.39	35.44	52.52	52.93	34.19	36.95	0.00	50.07	49.88
	CY	24.10	25.46	10.62	12.43	0.00	18.83	18.99	8.65	7.06	4.68	4.30	0.00	8.91	9.07	18.68	19.17	3.49	4.28	0.00	18.50	18.56
	BD	252.97	242.18	121.24	121.70	0.00	162.33	160.50	89.07	88.79	54.61	55.06	0.00	98.56	97.59	145.50	143.91	87.62	88.73	0.00	140.94	139.07
	RE	−25.10	−19.60	−13.40	−11.77	0.00	−11.36	−9.94	−13.38	−12.03	−8.76	−8.92	0.00	−3.93	−2.90	−5.80	−5.29	−11.75	−11.47	0.00	−5.20	−4.64
	$\mathcal{C}_\tau$	14.93	14.89	9.30	9.29	2.72	11.30	11.30	8.23	8.22	5.93	5.93	2.72	7.83	7.83	10.75	10.75	7.75	7.73	2.72	10.54	10.54
	$\mathcal{C}'_\tau$	14.89	14.89	9.27	−0.17		11.30	8.86	7.96	−10.99	5.62	−21.06		7.82	−5.84	10.75	6.63	7.48	−18.96		10.54	5.43

Note: For different investment horizons ($\tau \in \{1, 5, 10, 20\}$ in days) and for different combinations of preference parameters ($\gamma = 3$, $\ell \in [0, 3]$, $\kappa \in \{0.95, 0.975, 1, 1.025, 1.05\}$), the table displays estimates of the optimal portfolio weights (w) in individual risky assets (**SK** for stock, **CY** for commodity, **BD** for bond, and **RE** for real estate), and the associated certainty equivalent, $\mathcal{C}_\tau$, without the approximation (N) and with the approximation (Y). The cell $\mathcal{C}'_\tau$(N) is the certainty equivalent of the right-hand side of equation (19), while $\mathcal{C}'_\tau$(Y) corresponds to the certainty equivalent of the standard MV portfolio with equal DA-implied risk aversion. The asset menu comprises the risk-free rate and four indices including stock, commodity, bond, and real estate. The data are daily, and the sample period is from January 2, 1989 to October 31, 2022.



Note: For different investment horizons ($\tau \in \{1, 5, 10, 20\}$ days) and preference parameters ($\gamma = 3$, $\ell \in [0, 3]$, $\kappa \in \{0.95, 0.975, 1, 1.025, 1.05\}$), Panel A shows estimates of $\lambda_{\mathcal{D},\tau}$, the weight assigned to the downside mean-variance certainty equivalent in the WAMV portfolio optimization of equation (15). Panel B displays estimates of $\widetilde{SR}_{w,\tau}$, the endogenous Sharpe ratio of the optimal portfolio. Panel C presents estimates of the endogenous Sharpe ratios of the basic risky assets. The asset menu includes a risk-free rate and four indices (stock, commodity, bond, real estate), with daily data from January 2, 1989, to October 31, 2022.

Figure A1: **Optimal portfolio and assets performance measures**



Note: For different investment horizons ($\tau \in \{1, 5, 10, 20\}$ in days) and for different combinations of preference parameters ($\gamma = 3$, $\ell \in [0, 3]$, $\kappa \in \{0.95, 0.975, 1, 1.025, 1.05\}$), the figure displays in Panel A the estimates of $\alpha_{\mathcal{D},\tau}$, the weight assigned to the downside variance in computing the GDA-implied variance as in equation (14). In Panel B, it displays the estimates of $SR_{w,\tau}$, the standard Sharpe ratio of the optimal portfolio. The asset menu comprises the risk-free rate and four indices including stock, commodity, bond, and real estate. The data are daily, from January 2, 1989 to October 31, 2022.

Figure A2: **Optimal downside variance weight and standard Sharpe ratio**